Perspective

The Mathematics of Depicting Depth

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Colorado College

Fearless Fridays
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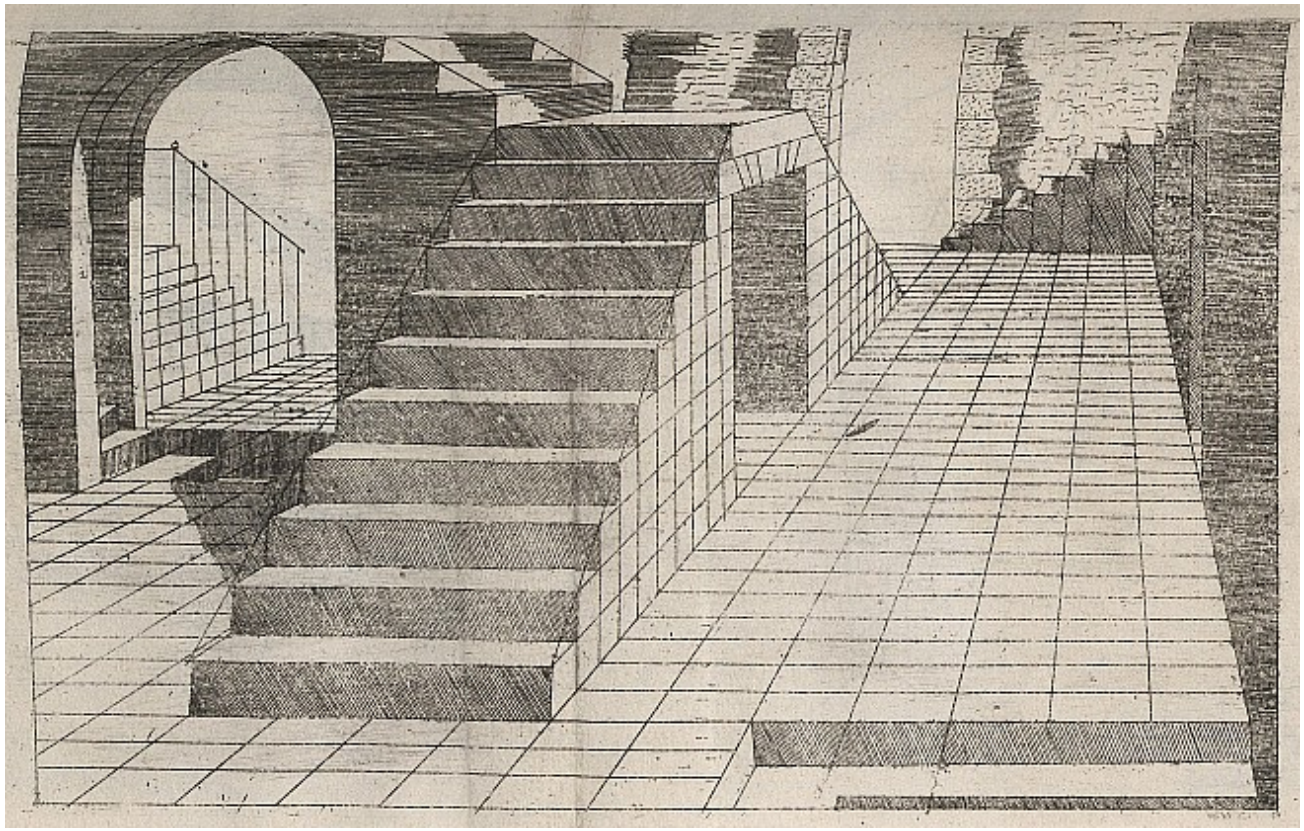
Outline

Introduction

In art, the word *perspective* refers to various techniques for depicting depth on a two-dimensional surface.



Stairs in a House (1672 CE) by Franz Liser.

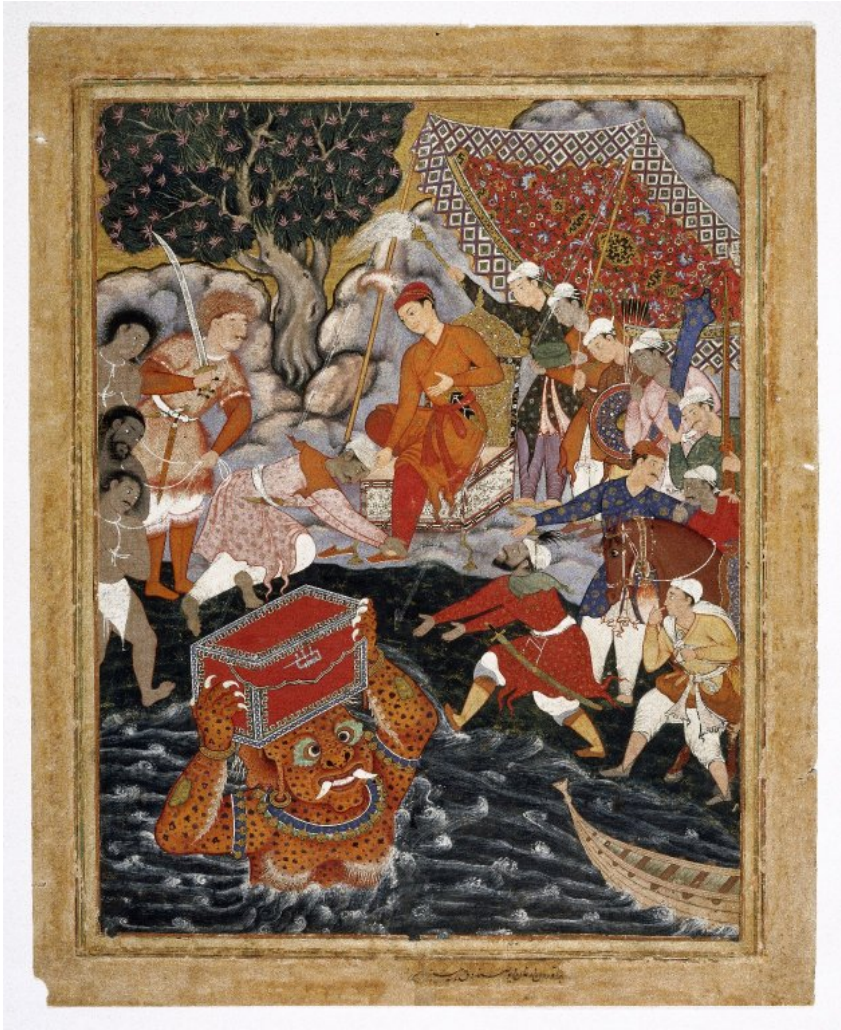
Introduction

Healing of the Cripple and Raising of Tabitha (1424 CE) by Masolino da Panicale.

Introduction

Cloudy Mountains (1130 CE) by Mi Youren (米友仁).

Introduction

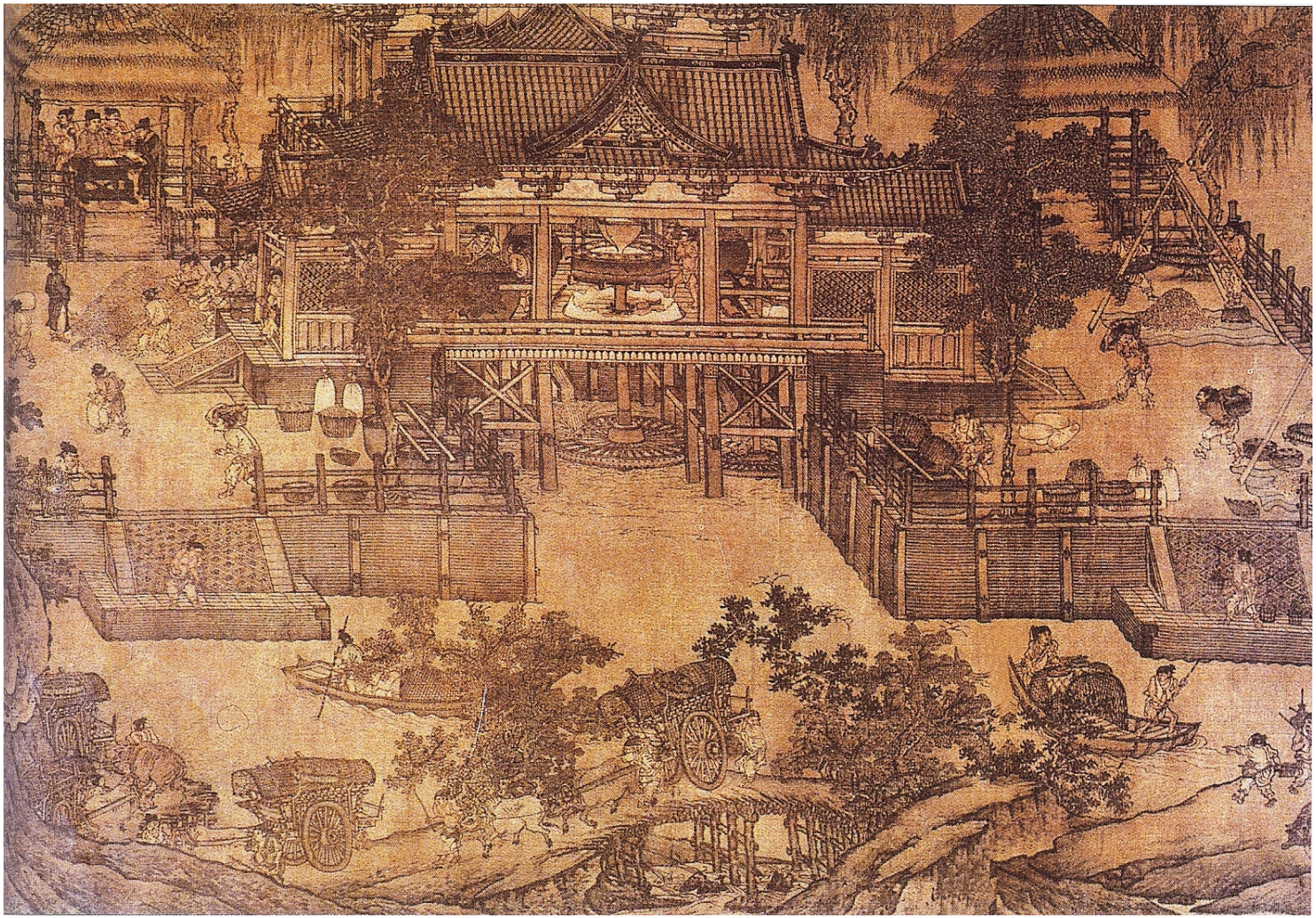


Painting from the *Hamzanama* (حمزه نامه, c. 1562–1577 CE), commissioned by Akbar.

Introduction

Wall painting from the Villa of P. Fannius Synistor at Boscoreale, near Pompeii (before 76 CE).

Introduction



Painting from the Northern Song Dynasty era (960–1127 CE).

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Painting by Shiba Kōkan (司馬 江漢, 1747–1818 CE).

Projection

First off, let's make a mathematical definition that we'll come back to a few times.

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Definition

A *projection* is a function from a subset of $\mathbb{R}^3$ to I, where I is a plane inside $\mathbb{R}^3$ called the *image plane*.

Outline

First Example

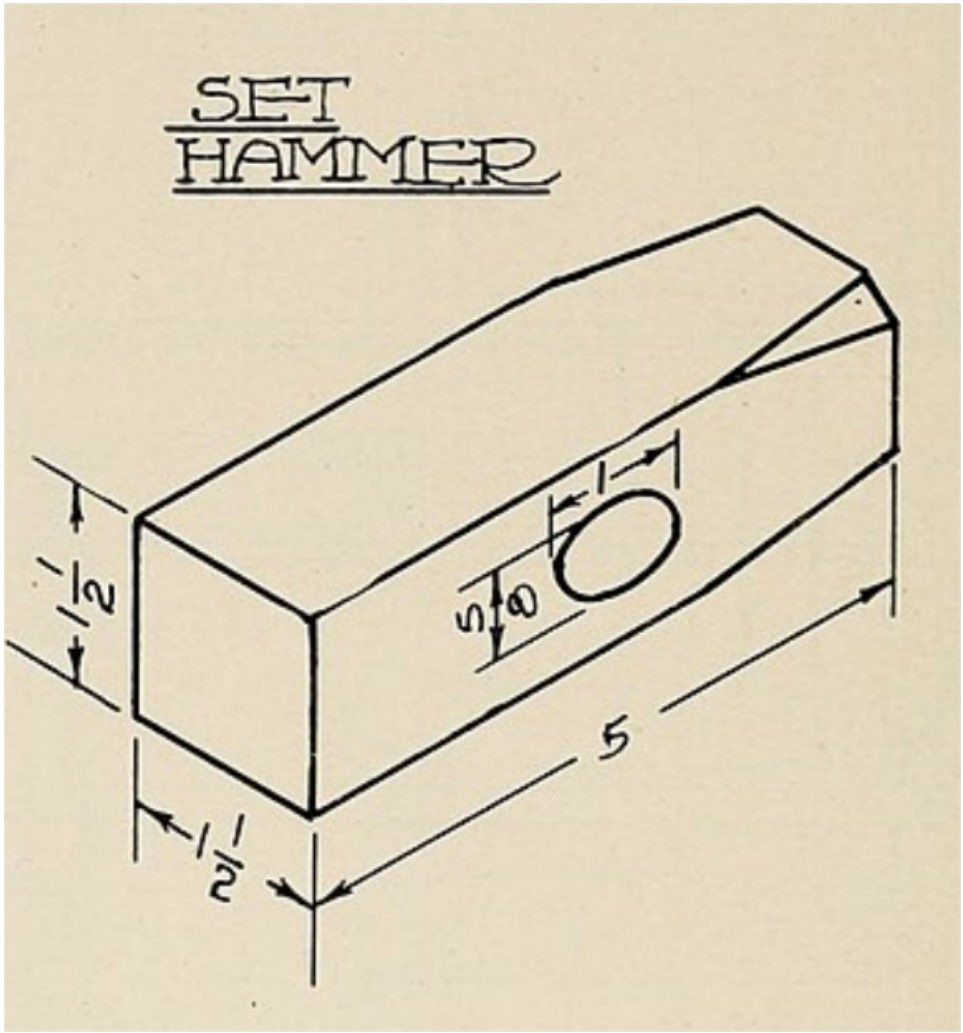
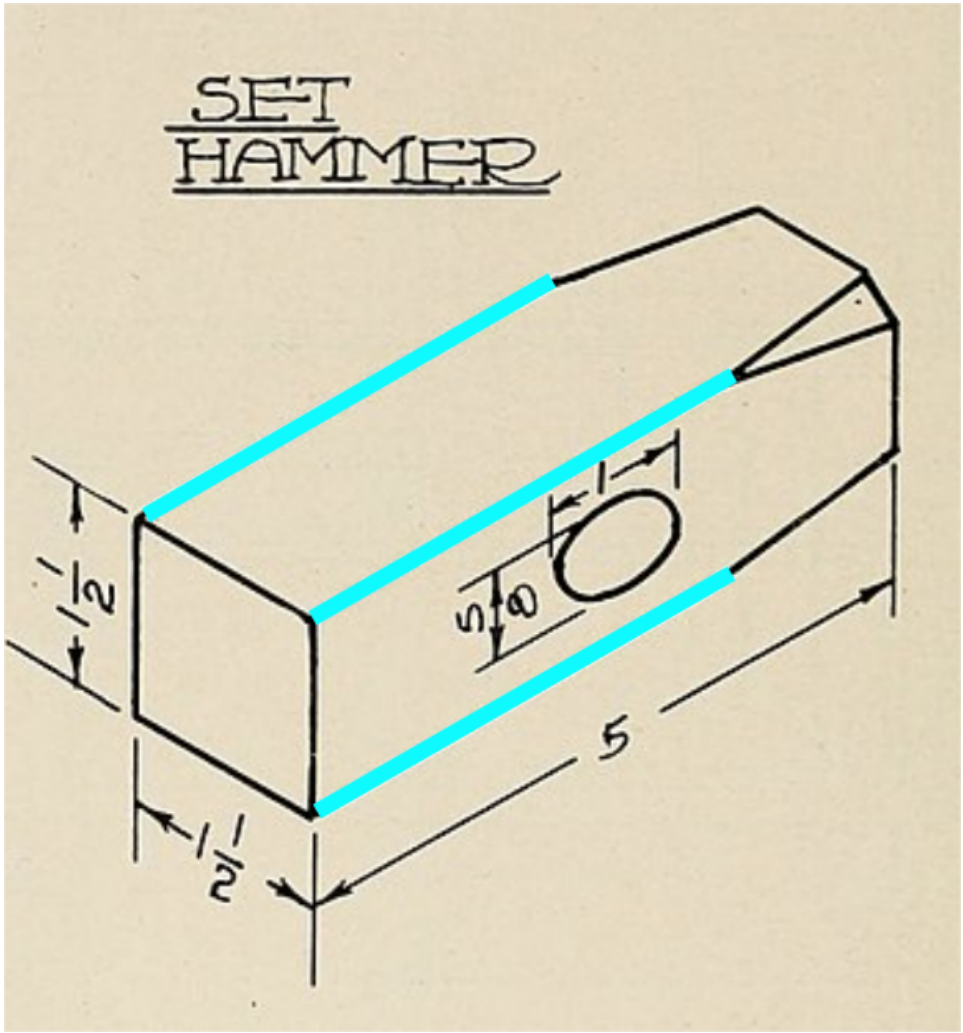


Diagram from a 1910 issue of *Industrial Education Magazine*.

First Example



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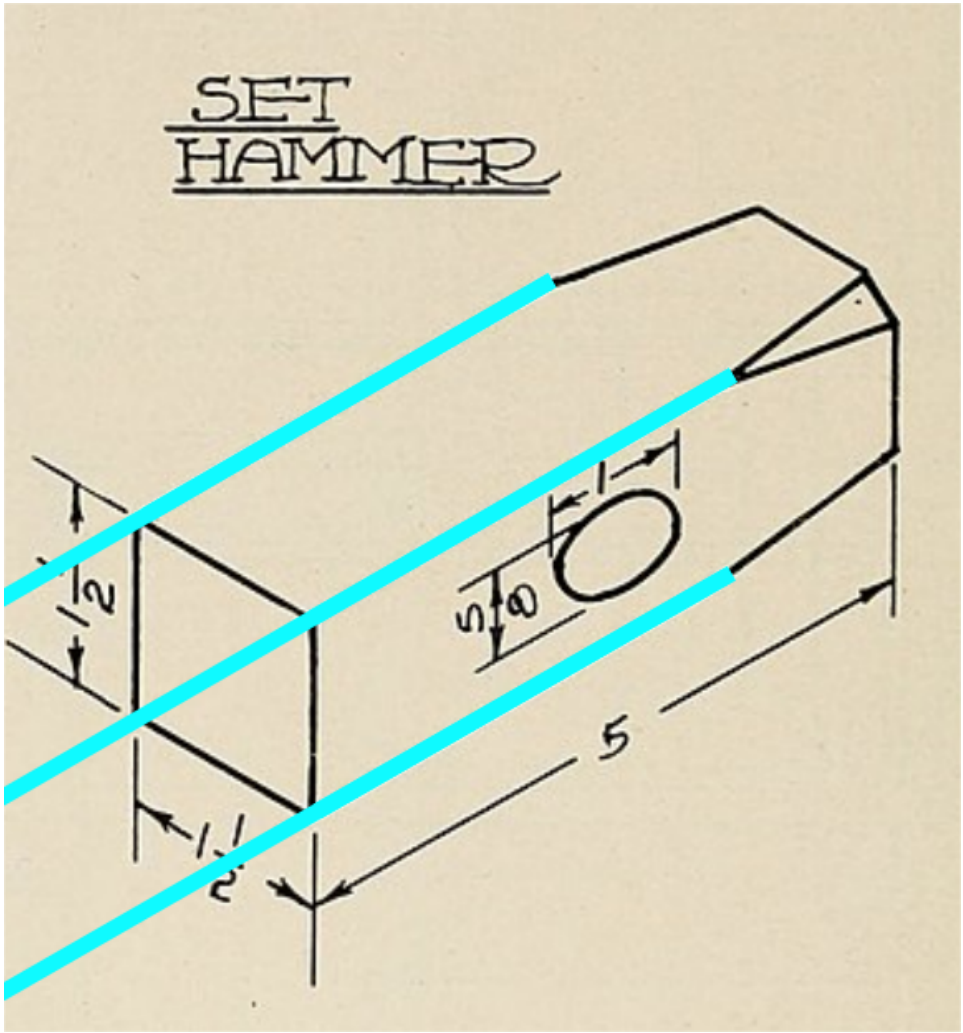

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Parallel Perspective

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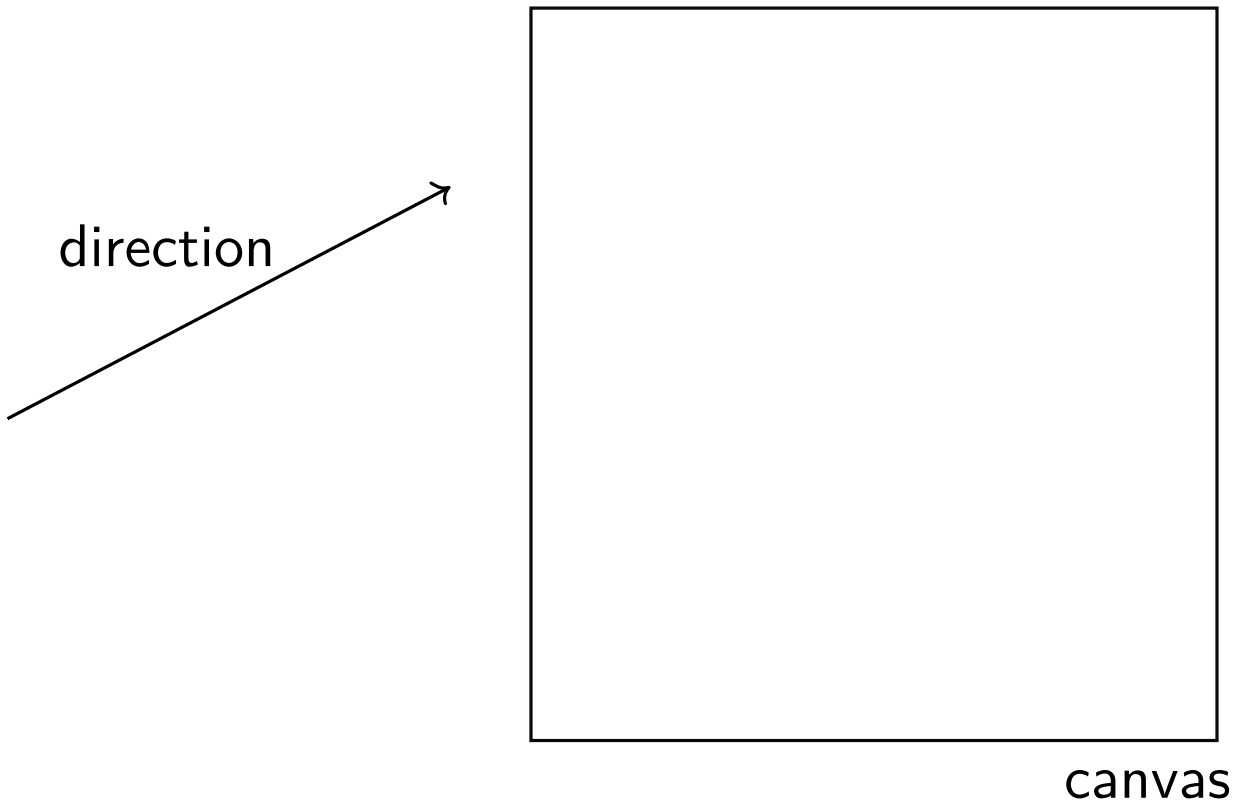
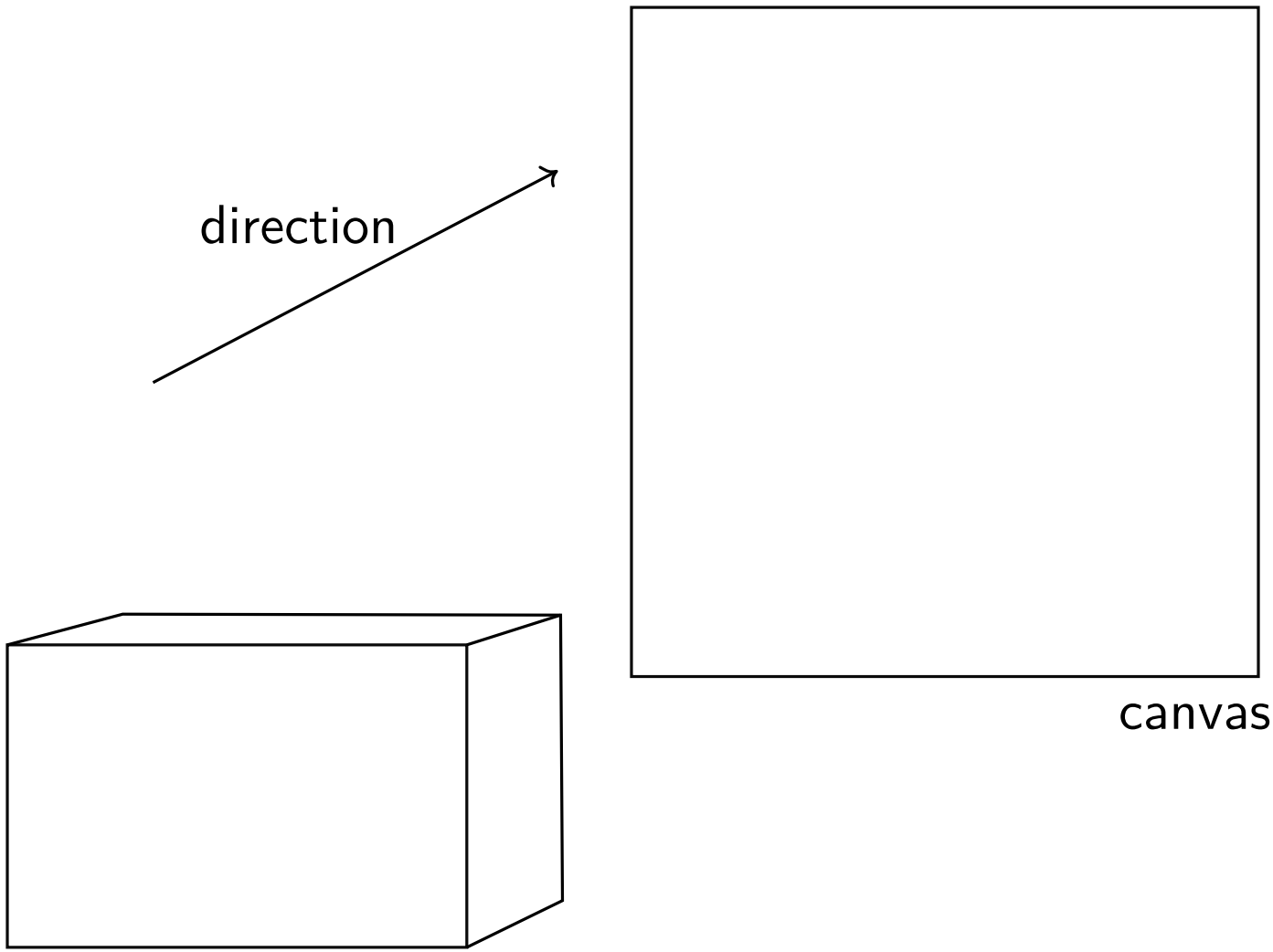
It can be modeled mathematically using a *parallel projection*.

Parallel Projection



Parallel Projection



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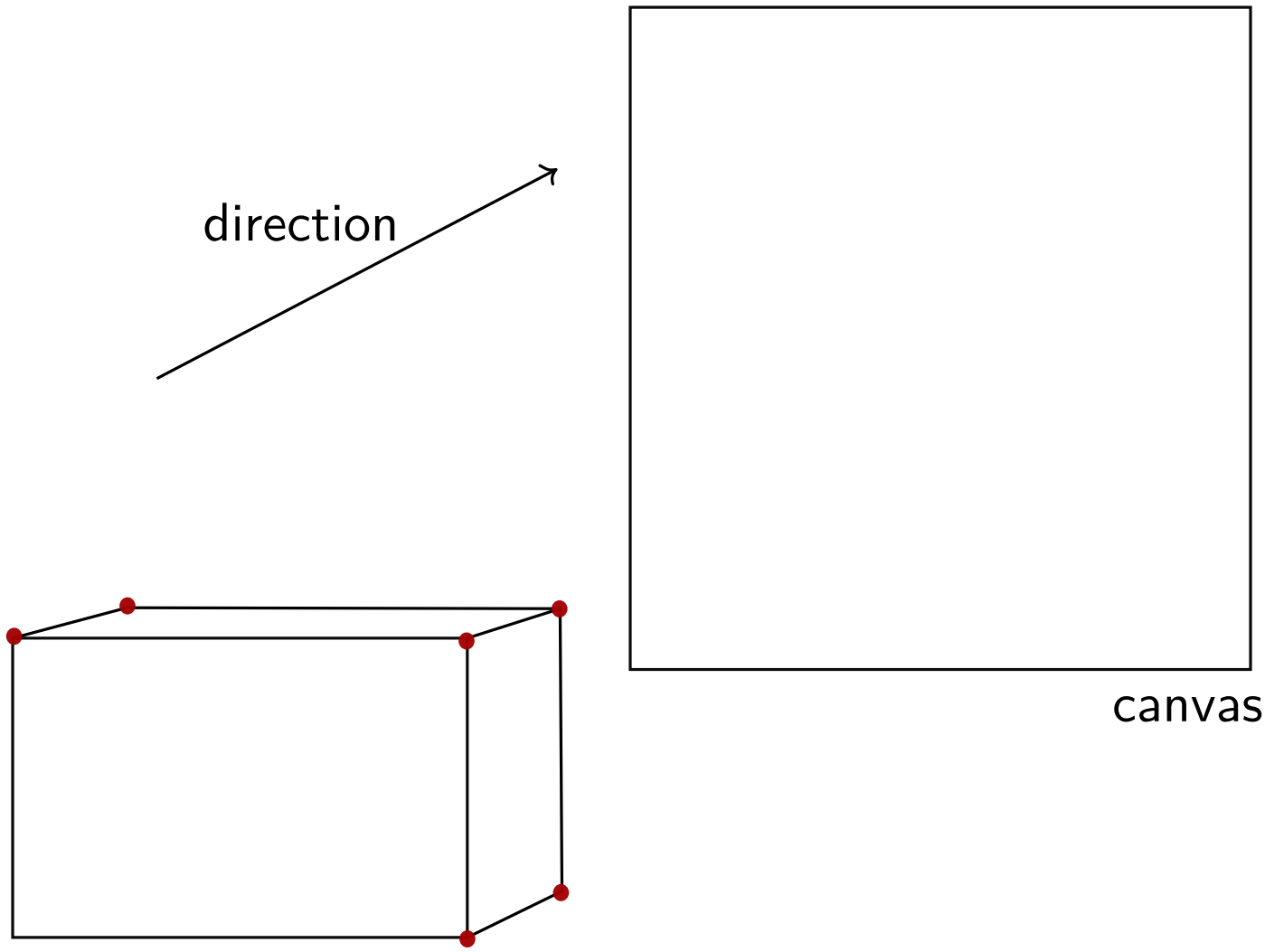

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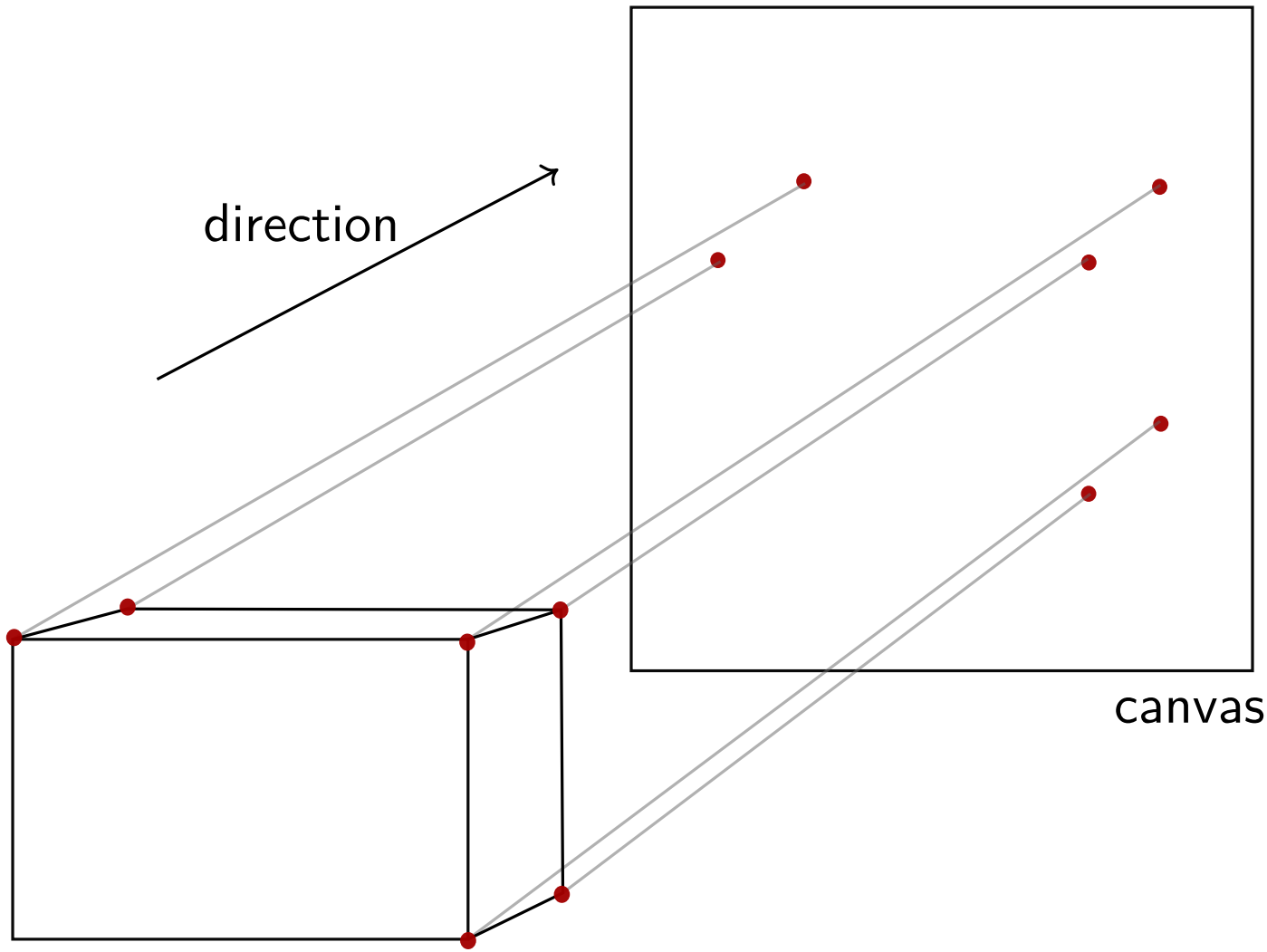

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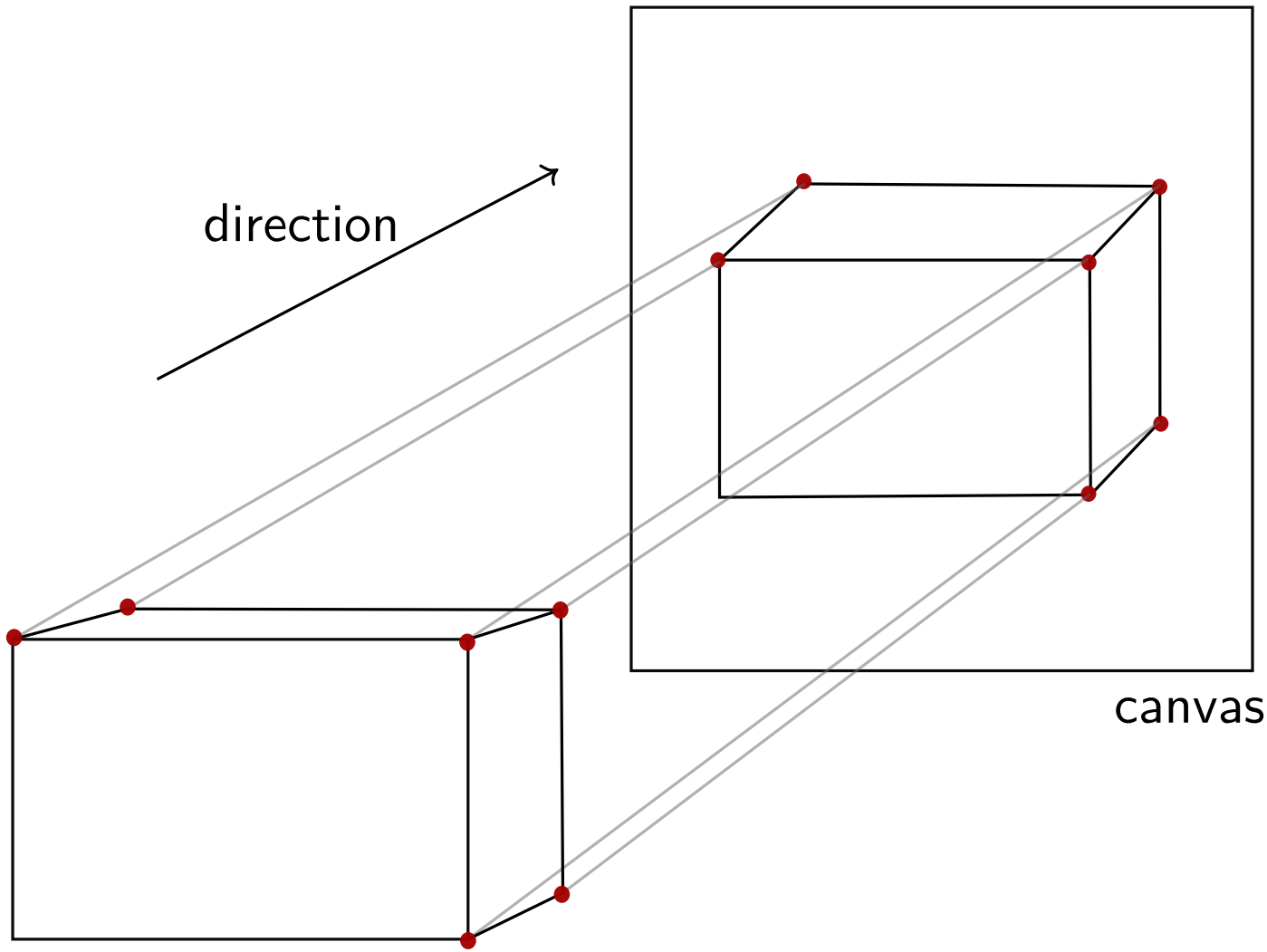

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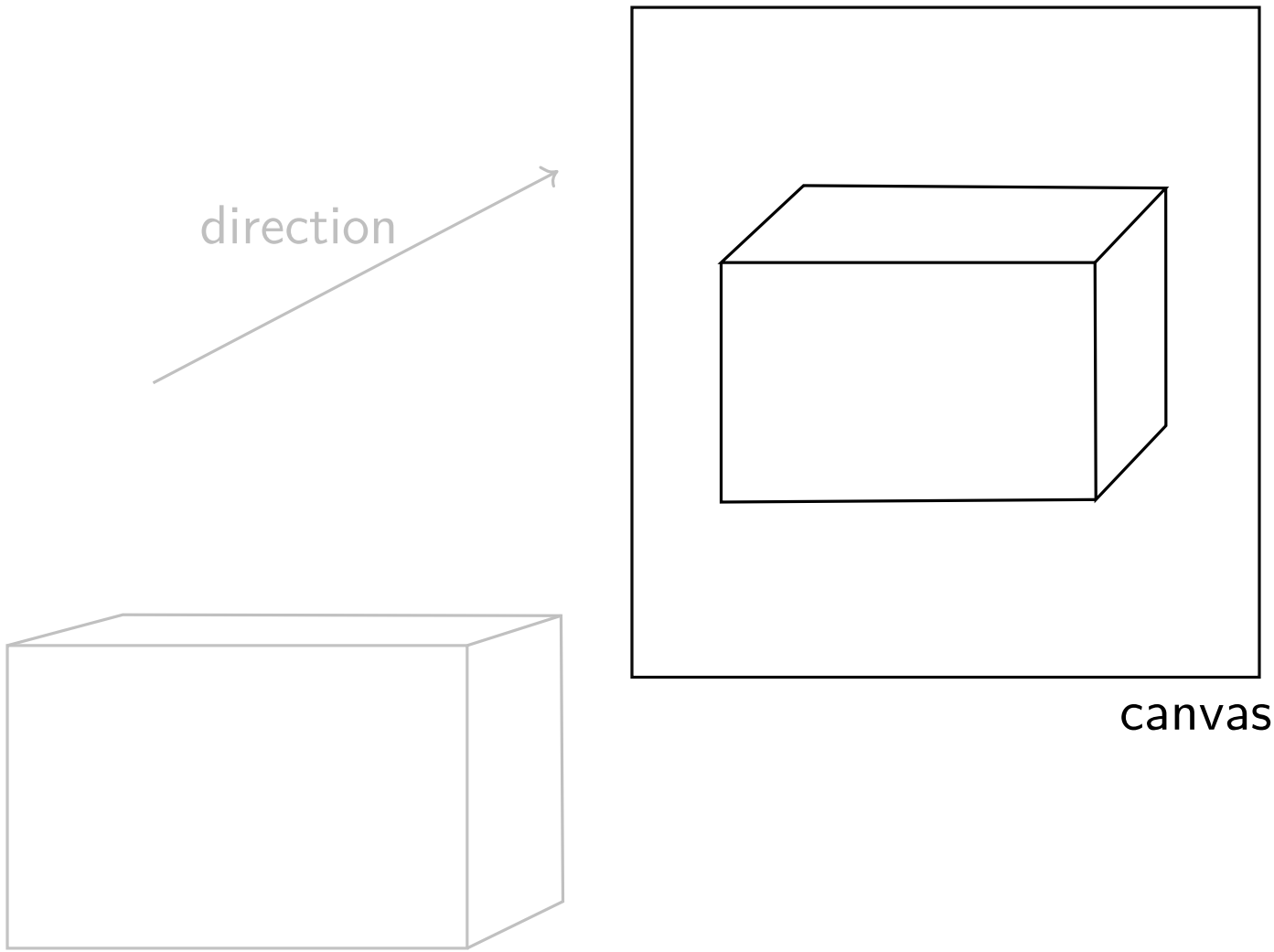

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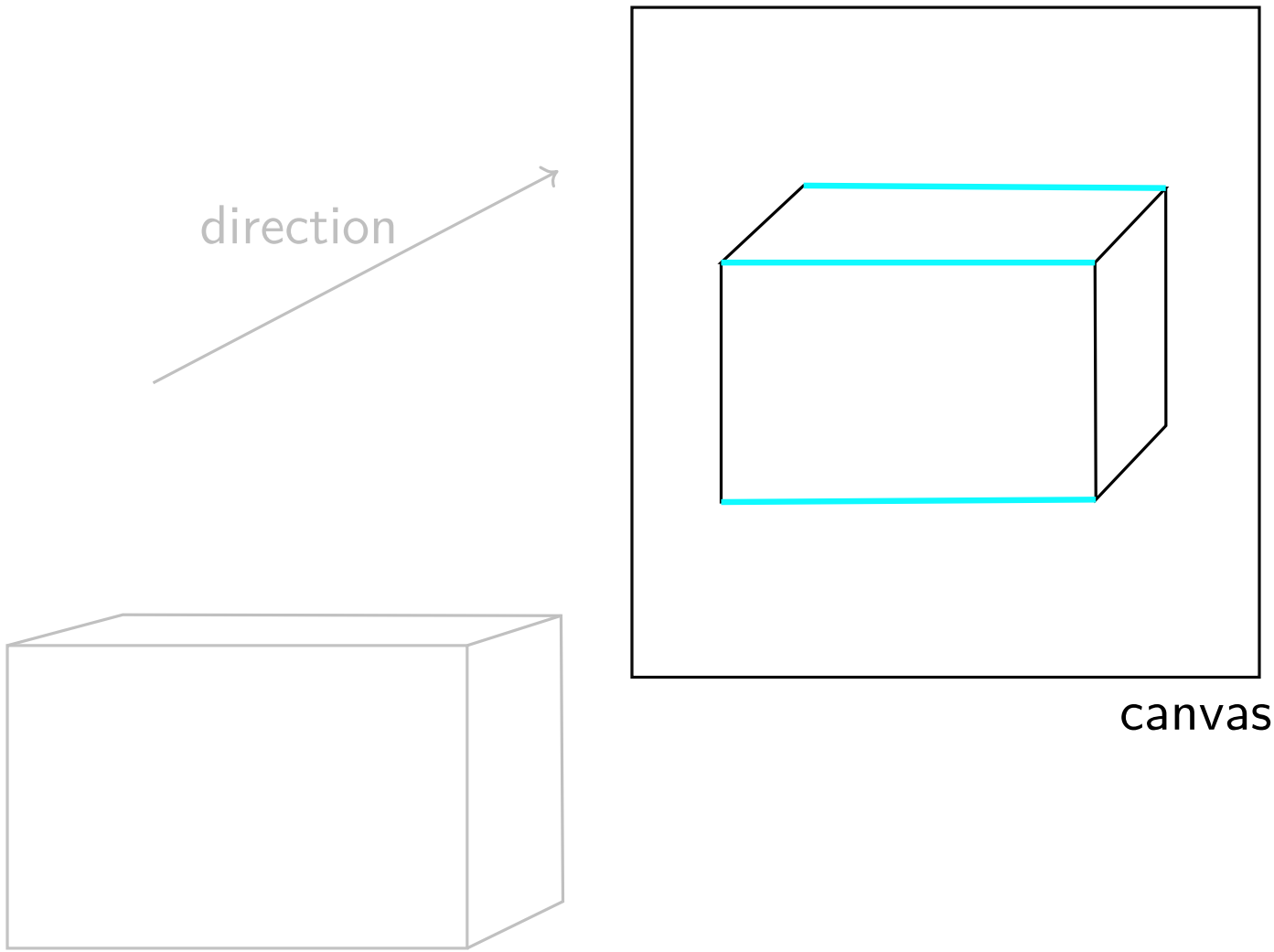

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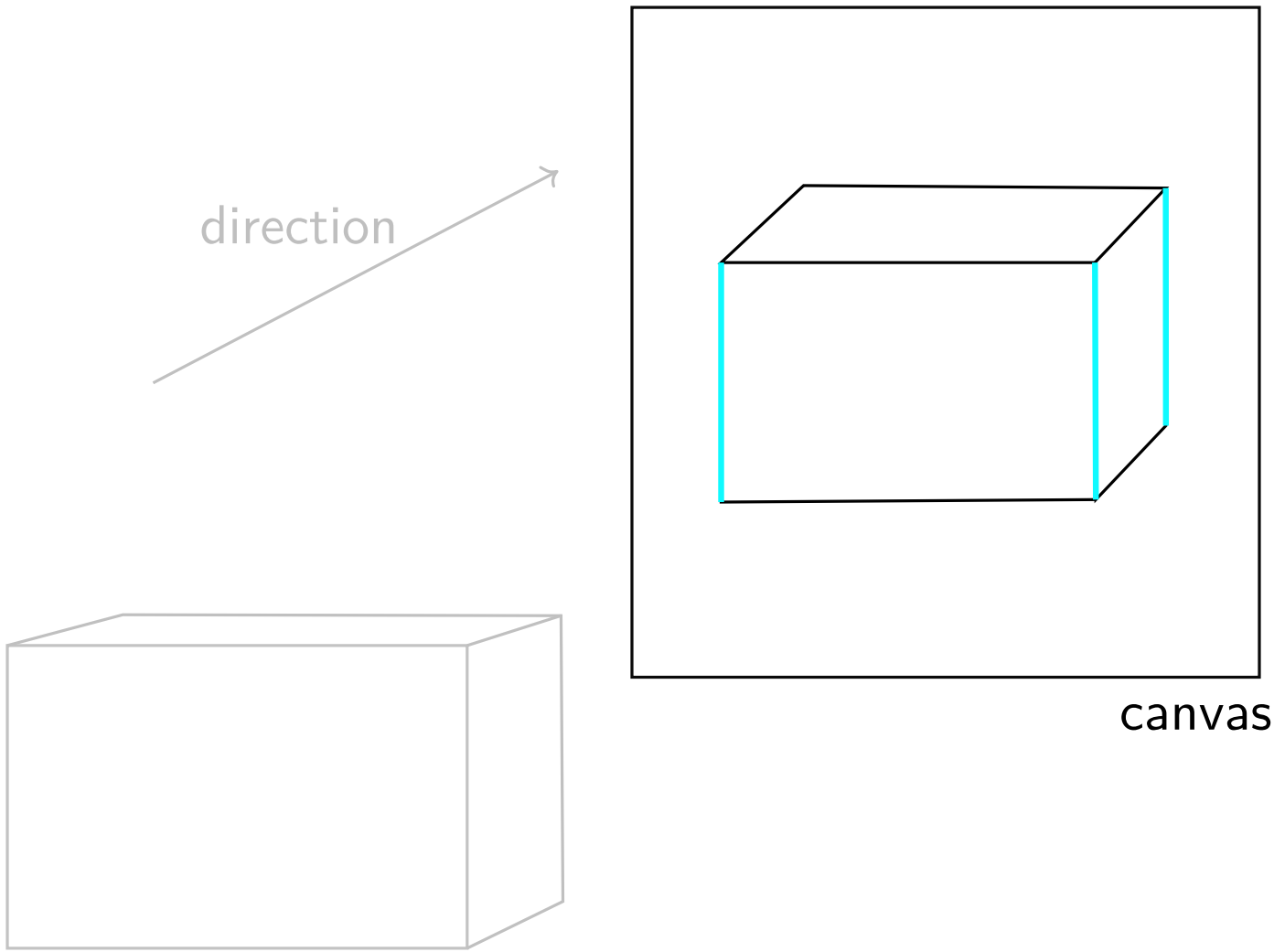

Parallel Projection

We can do this with any plane and any direction: if we choose a different image plane and/or a different direction in $\mathbb{R}^3$, we get a different parallel projection.

Naturality

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It shows up commonly in engineering drawings (in this case, the direction is often perpendicular to the image plane).

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It also shows up frequently and independently in art around the world.

Mughal India

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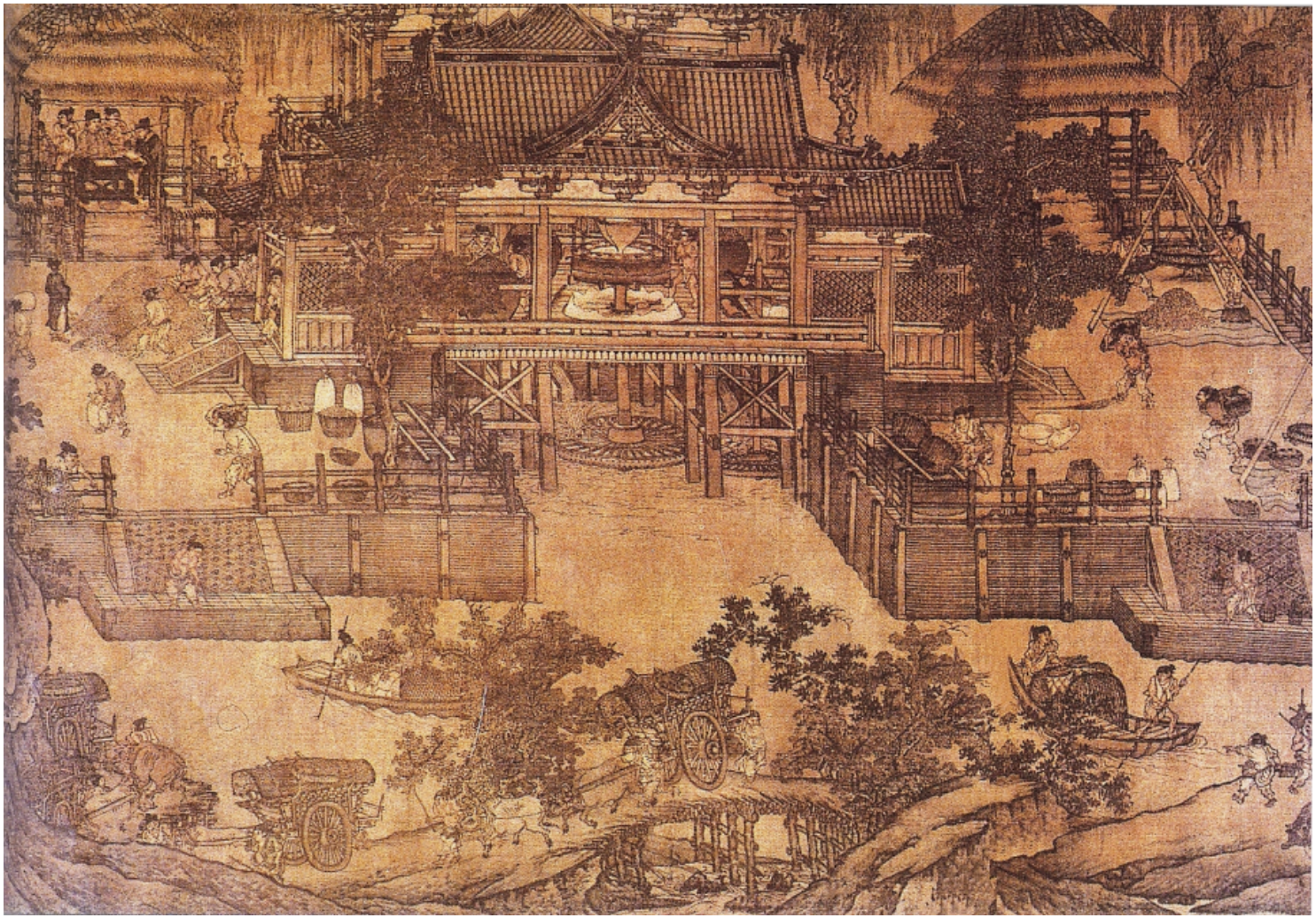
Song China



Painting from the Northern Song Dynasty era (960-1127 CE).

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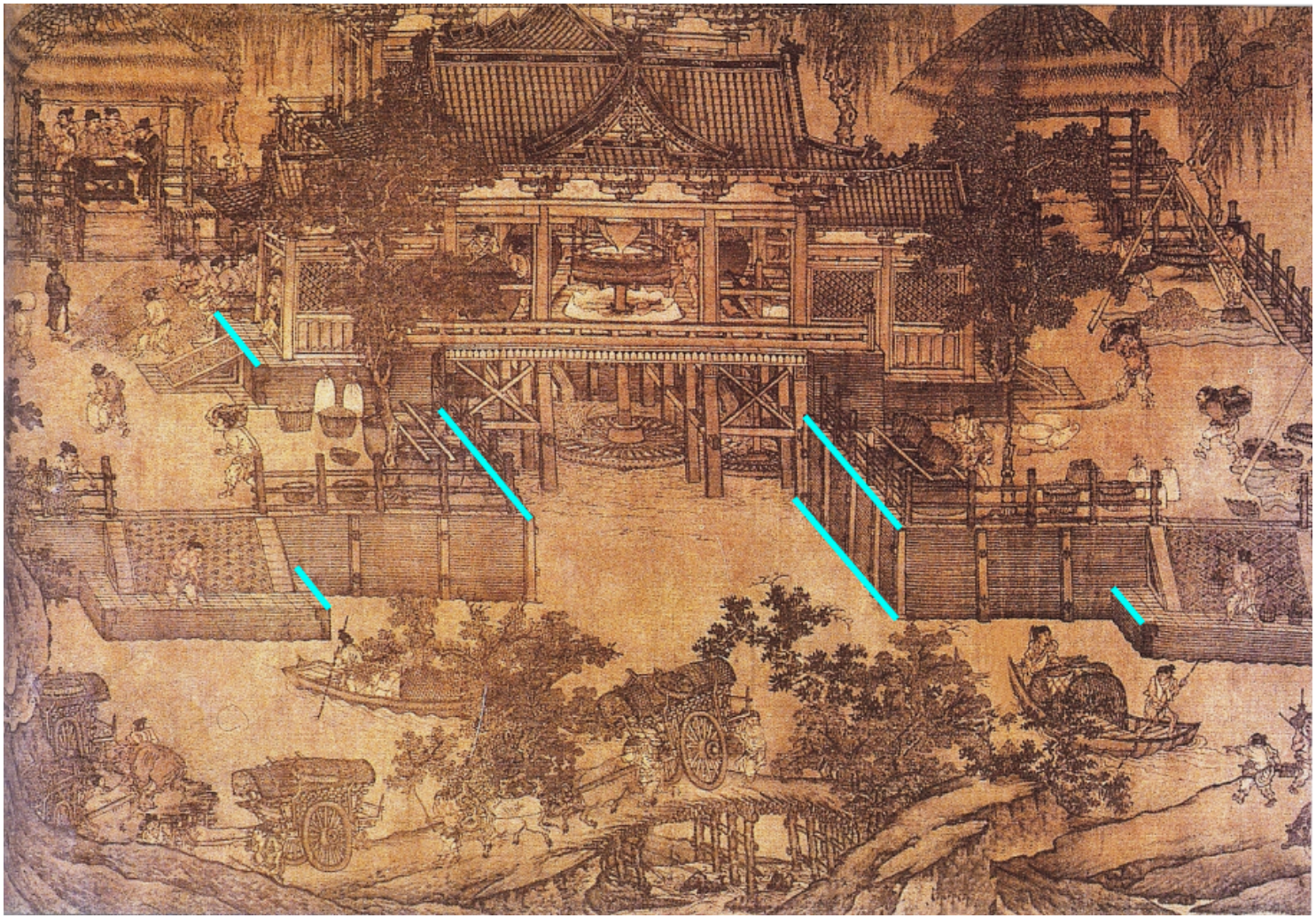

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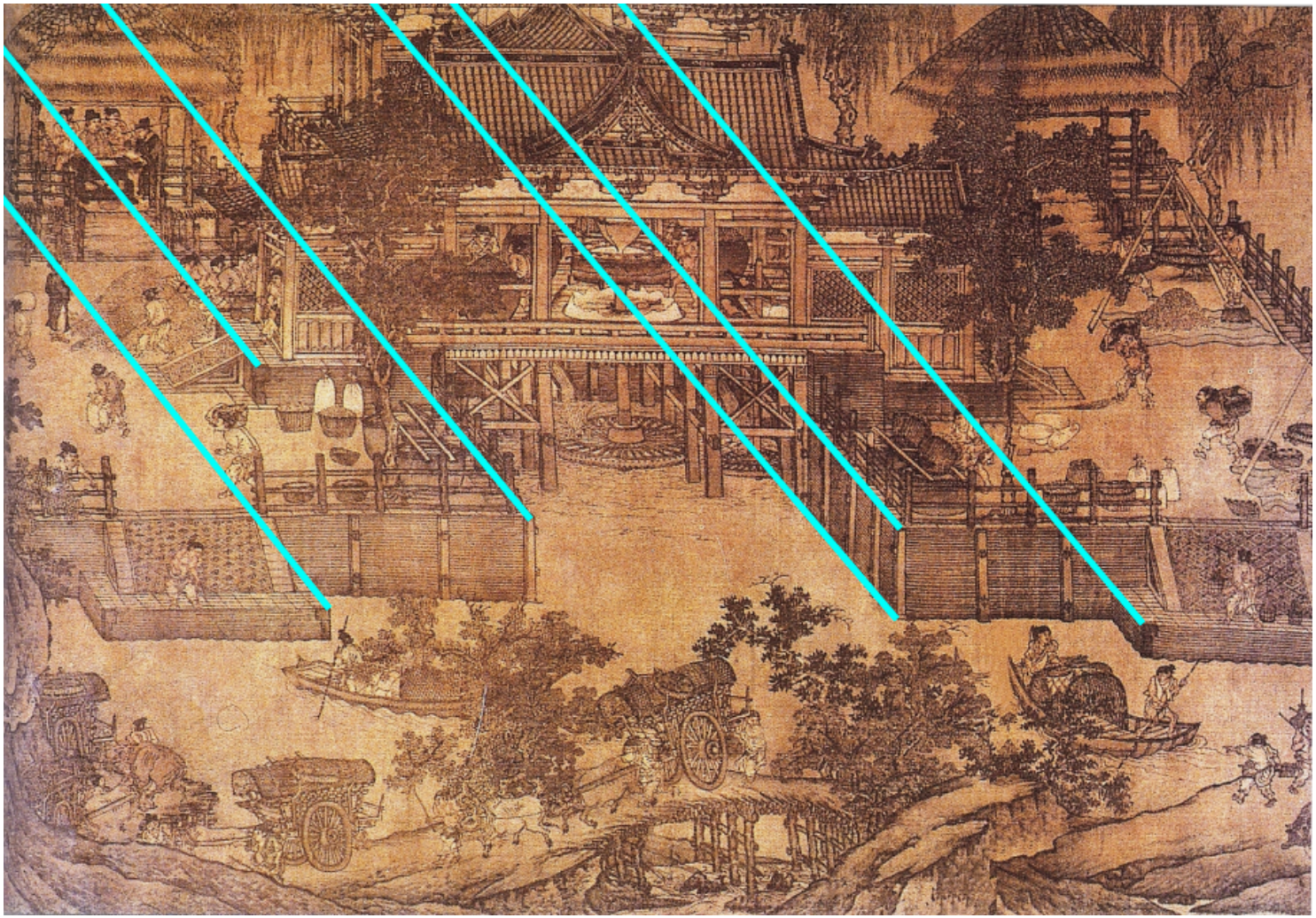

Painting from the Northern Song Dynasty era (960-1127 CE).

(Aside) Atmospheric Perspective

Cloudy Mountains (1130 CE) by Mi Youren (米友仁).

Outline

First Example

Railroad tracks in Lotus, Illinois.

Linear Perspective

In *linear perspective*, parallel lines in reality typically converge at a point in the image plane.

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Linear perspective can also be modeled by a projection, called a *perspective projection*.

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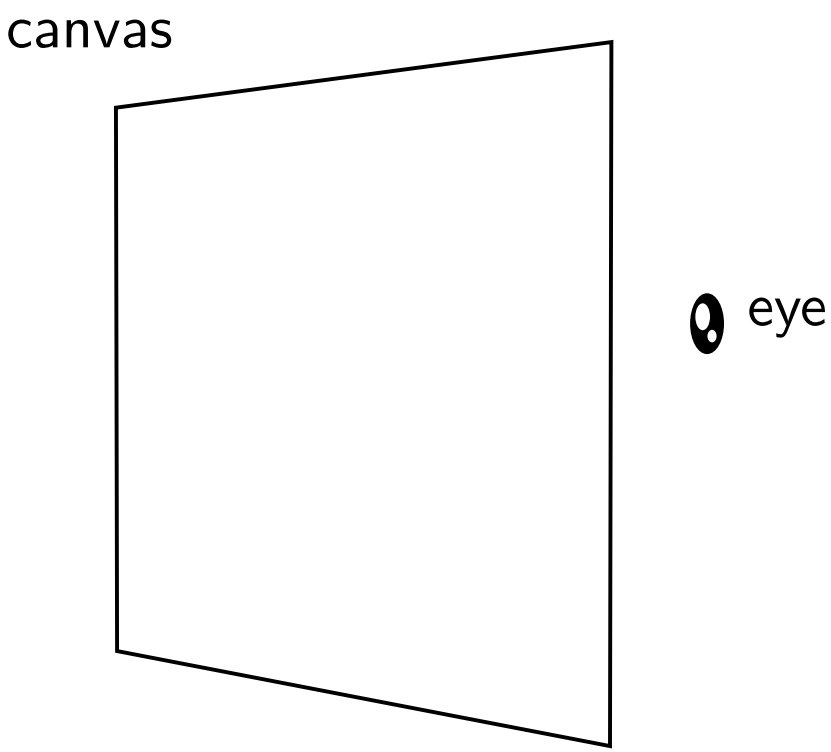
1. One way is as a model for sight.
2. The other way is as a model for drawing.

There's no significant mathematical difference between the two, but the drawing model is a little easier to visualize, so we'll think about that one first.

Perspective Projection



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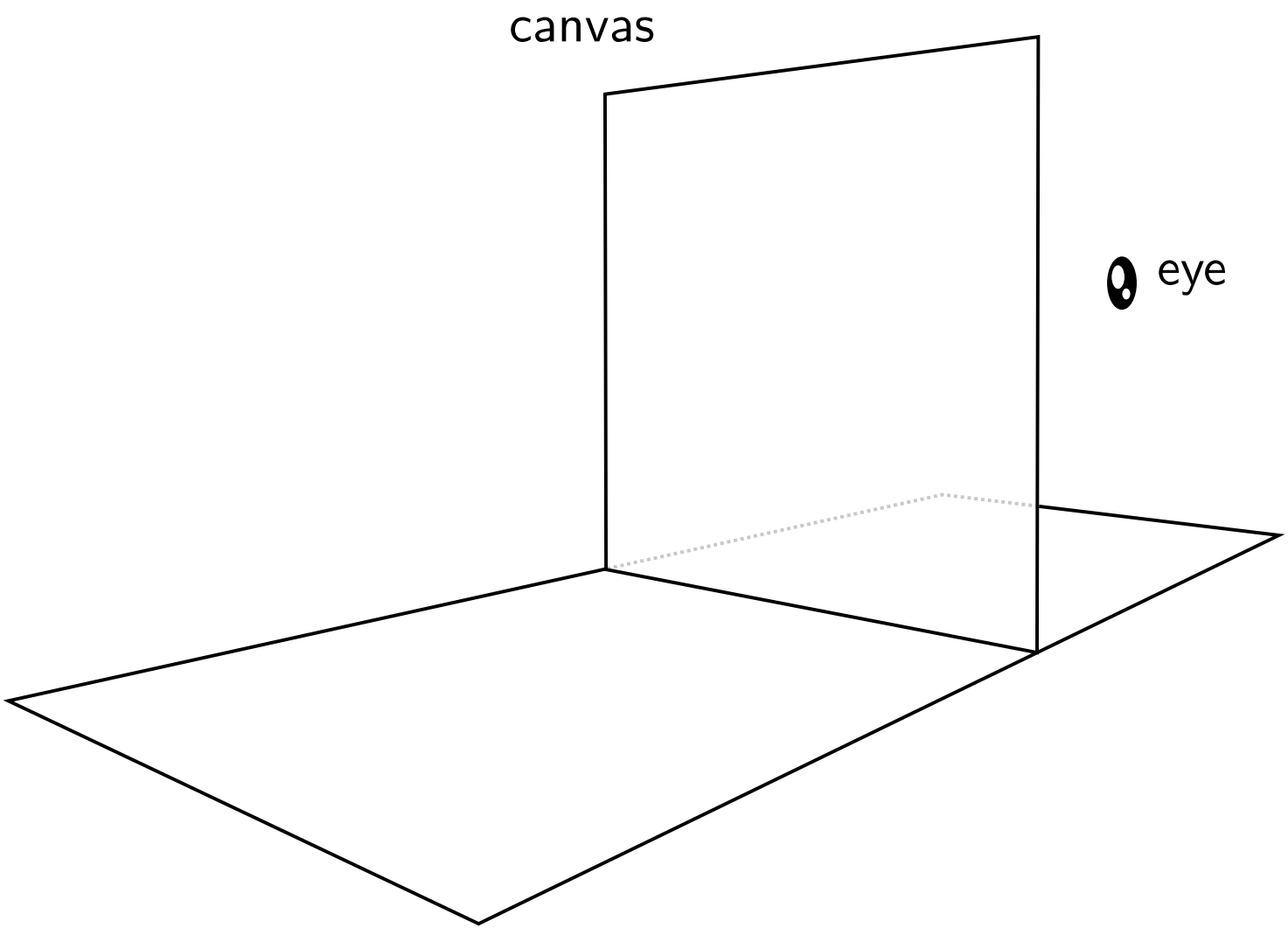

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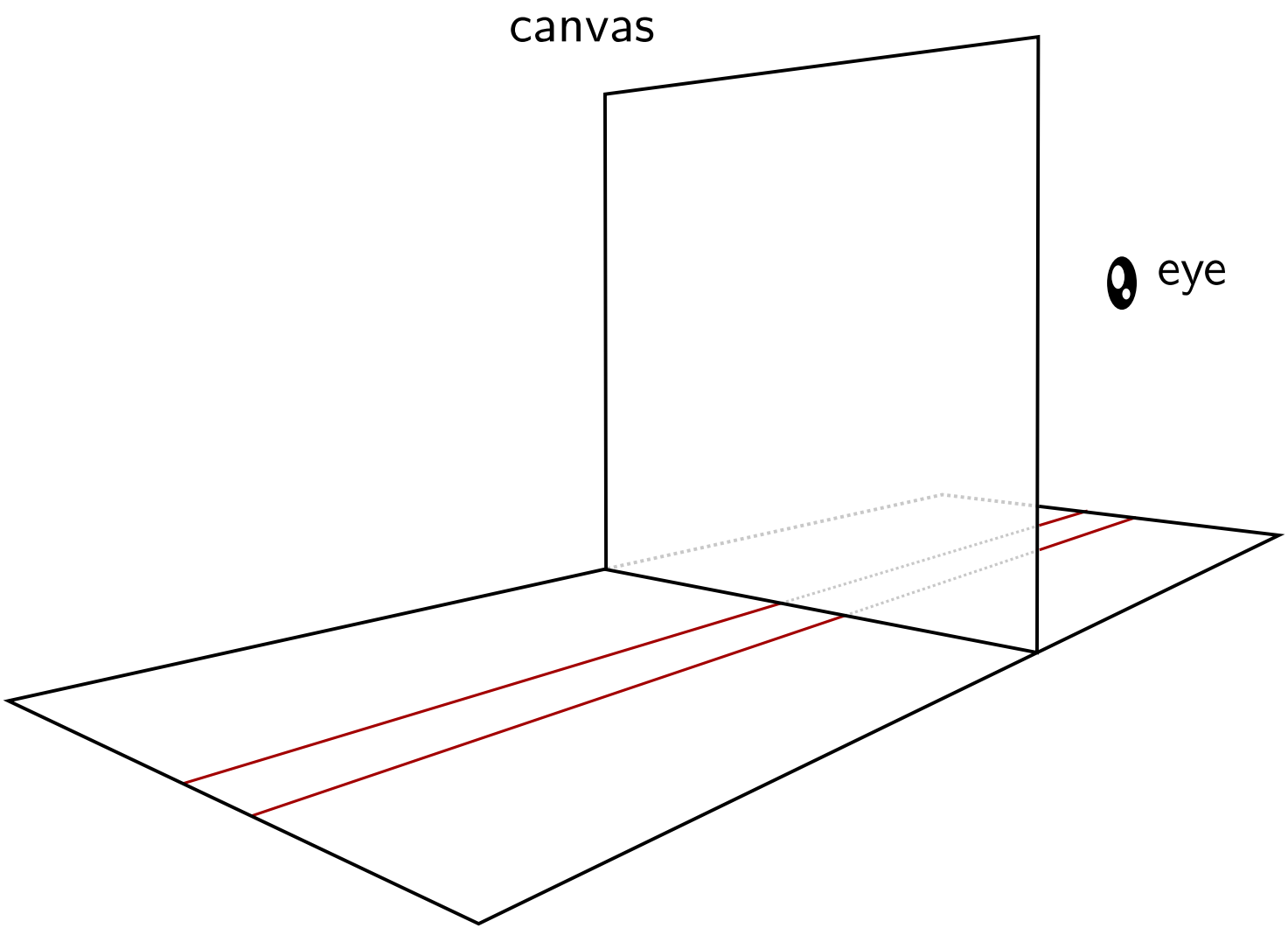

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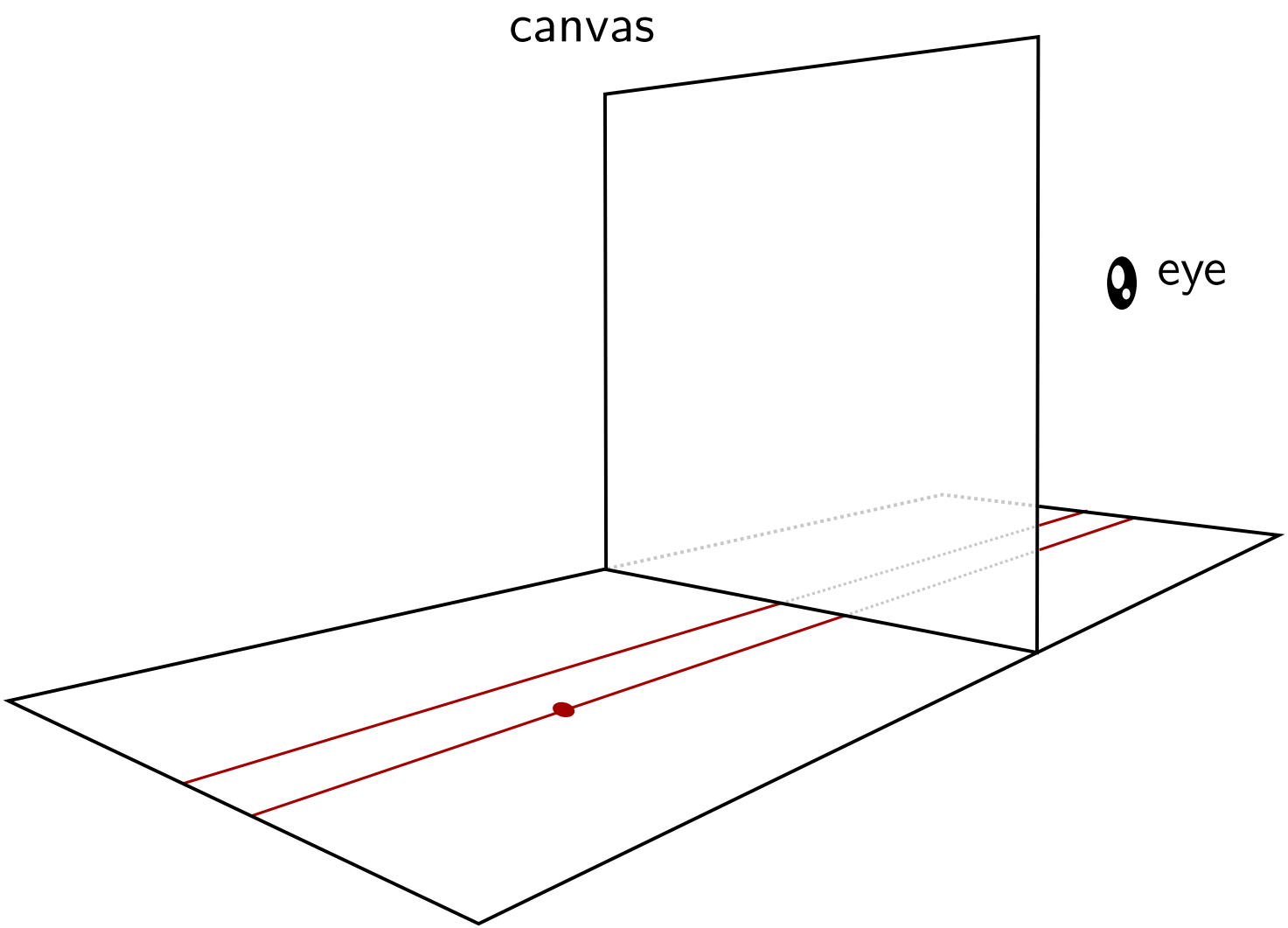

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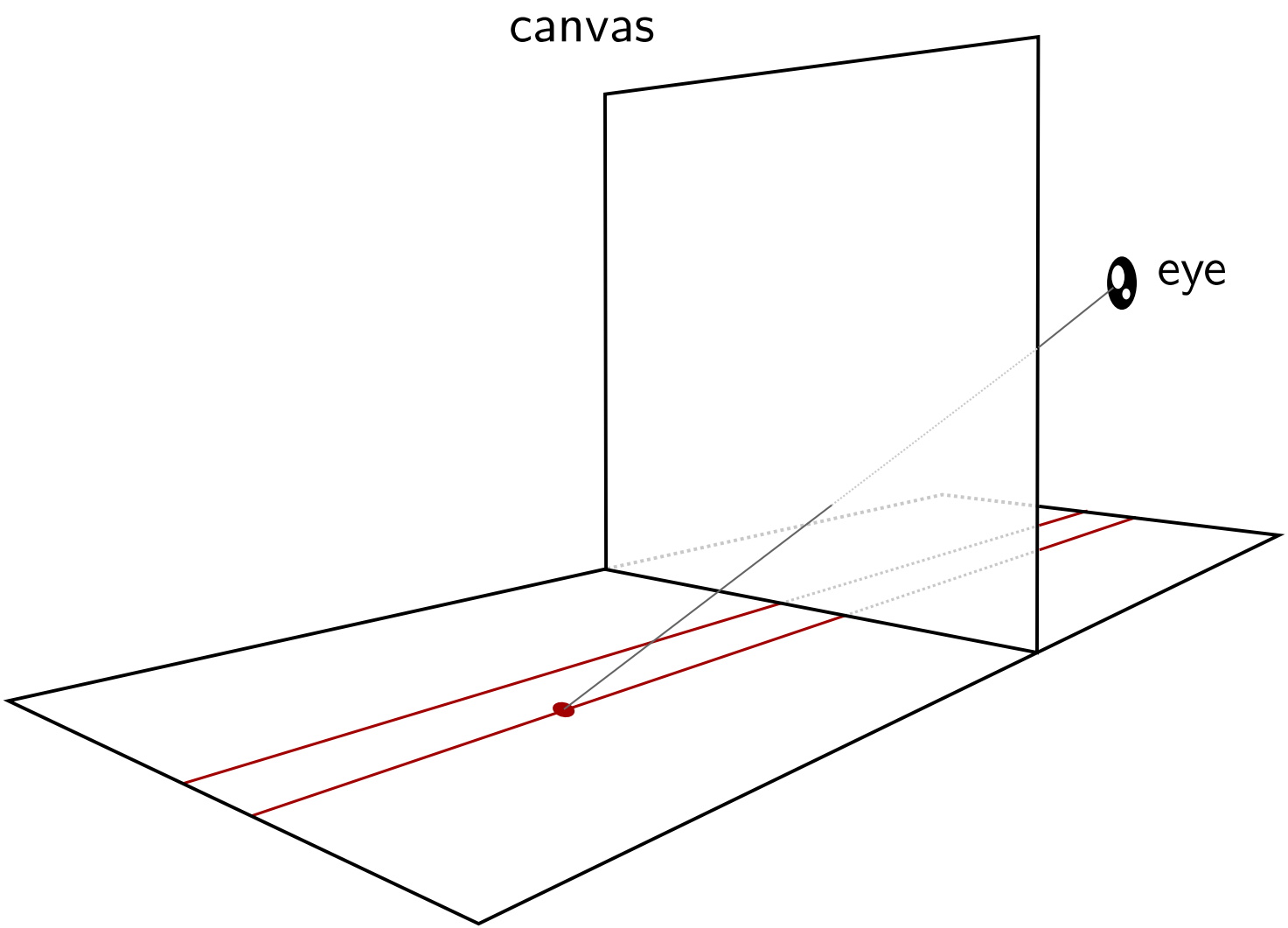

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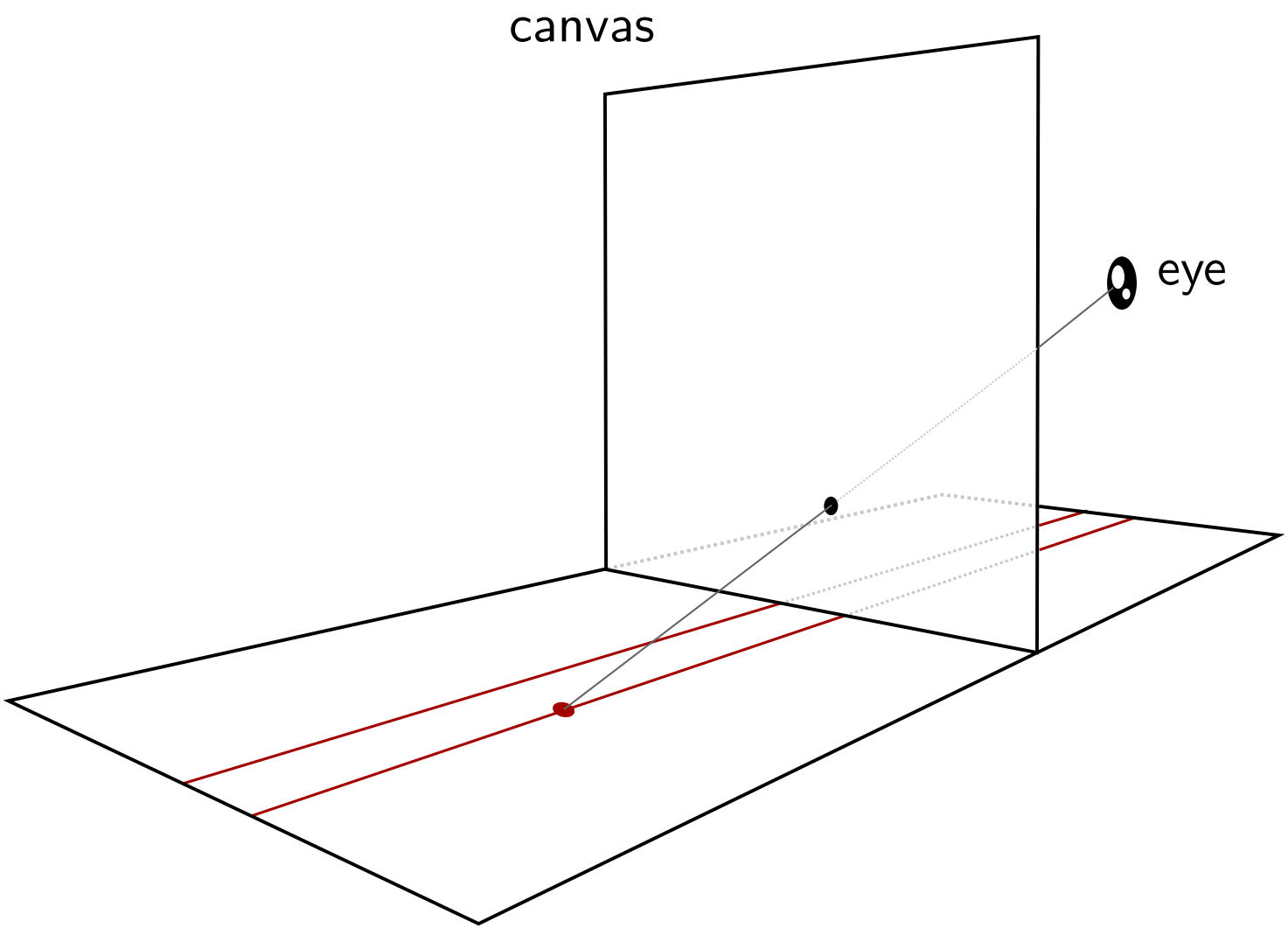

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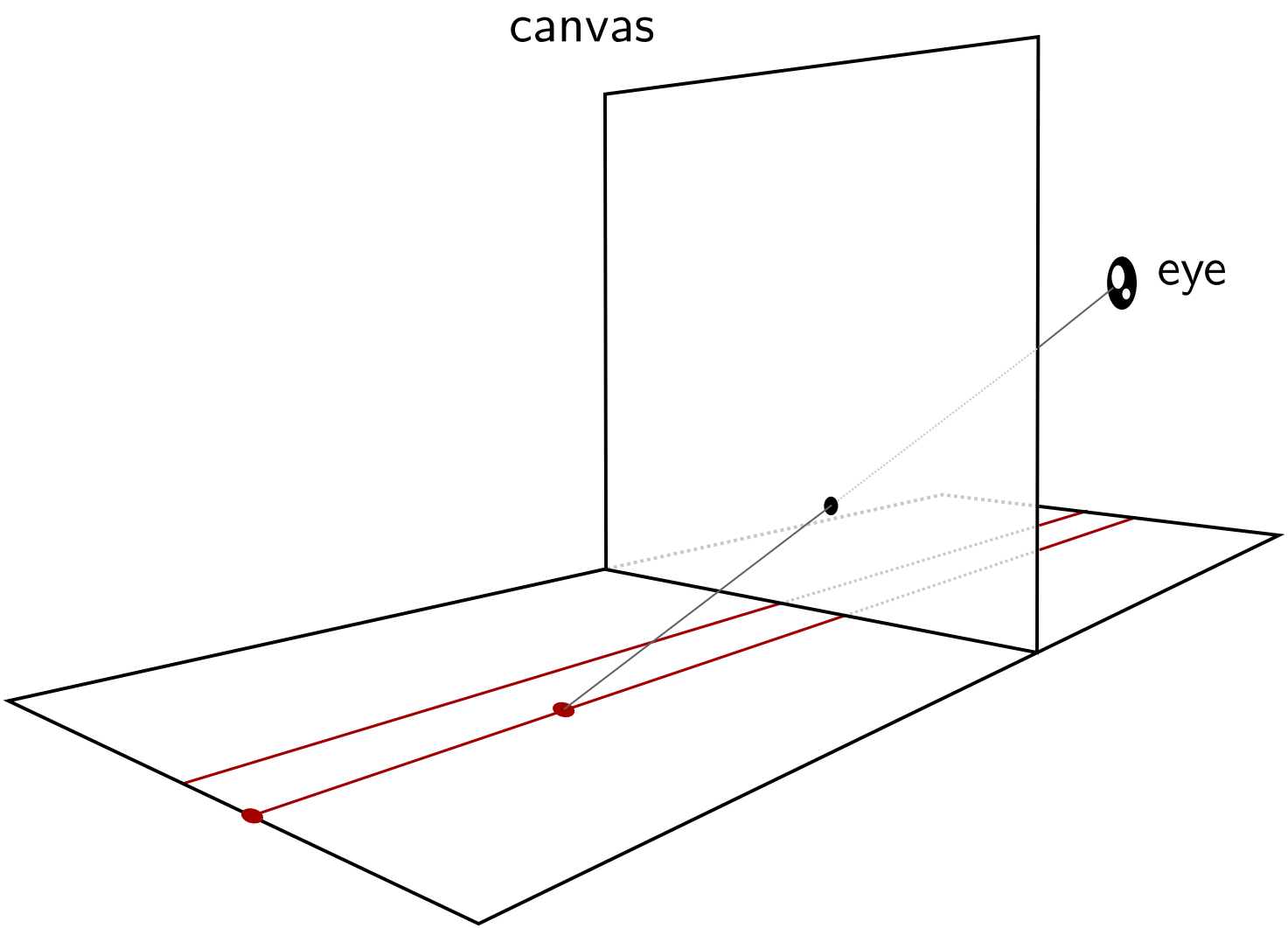

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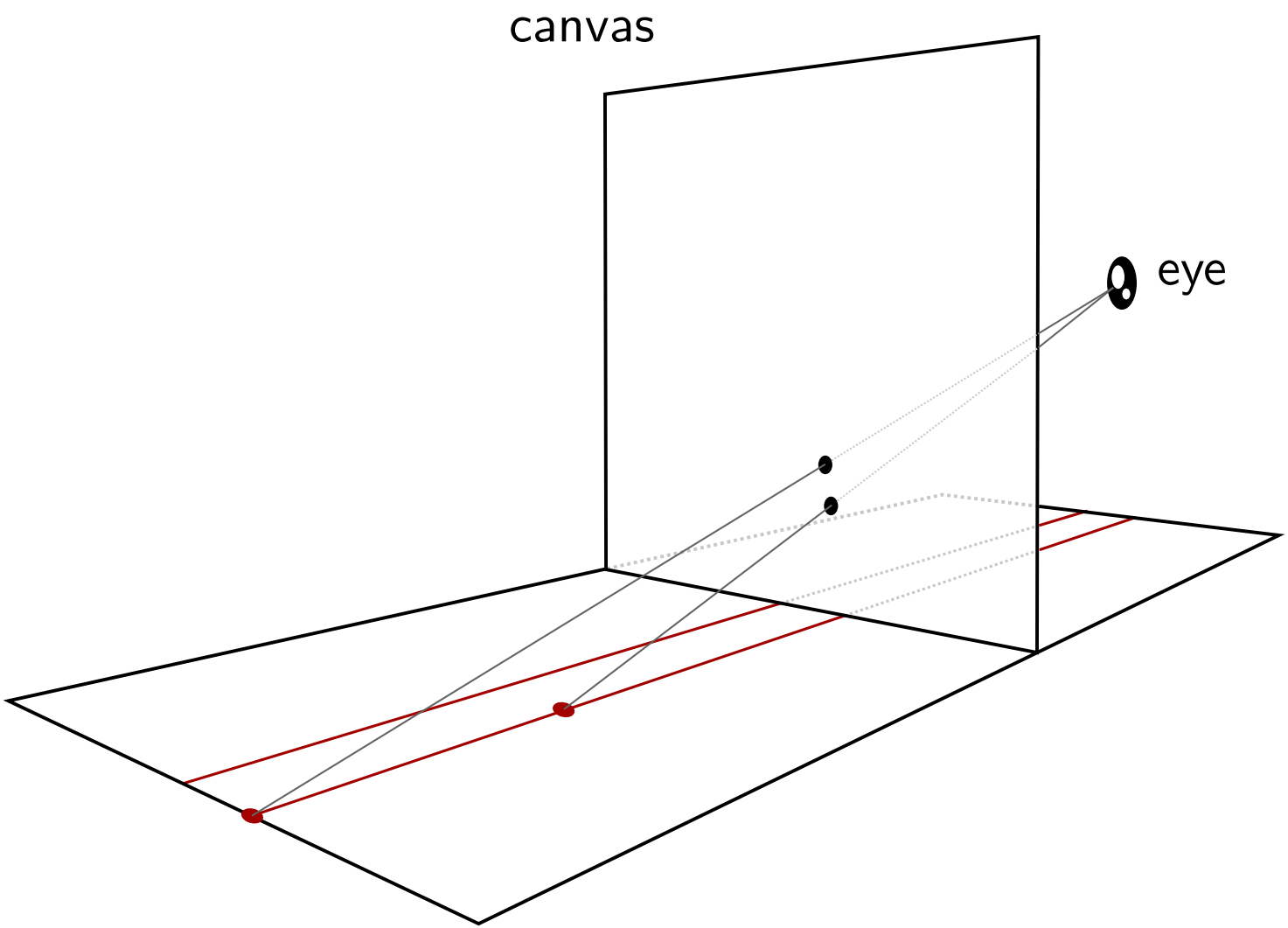

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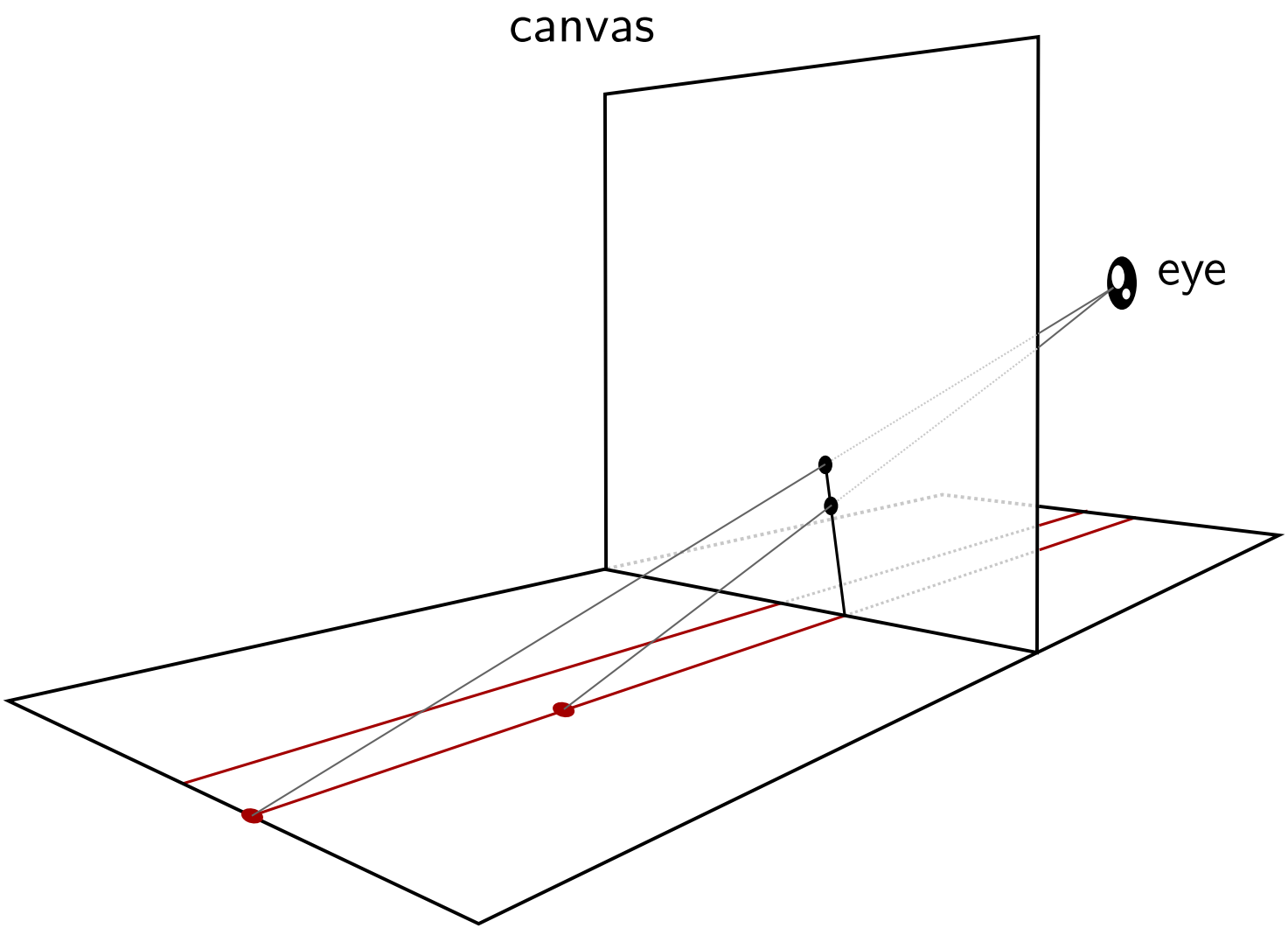

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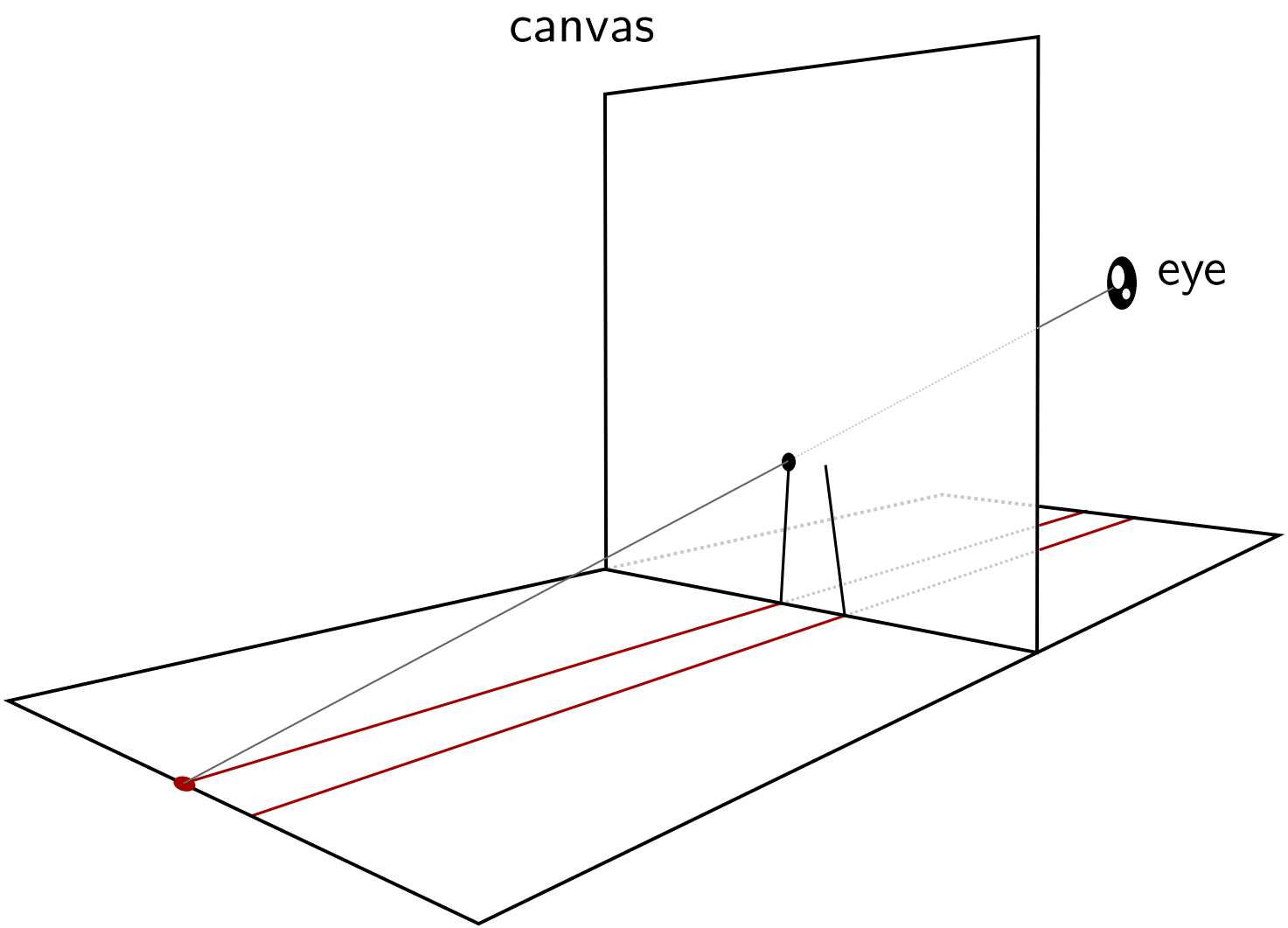

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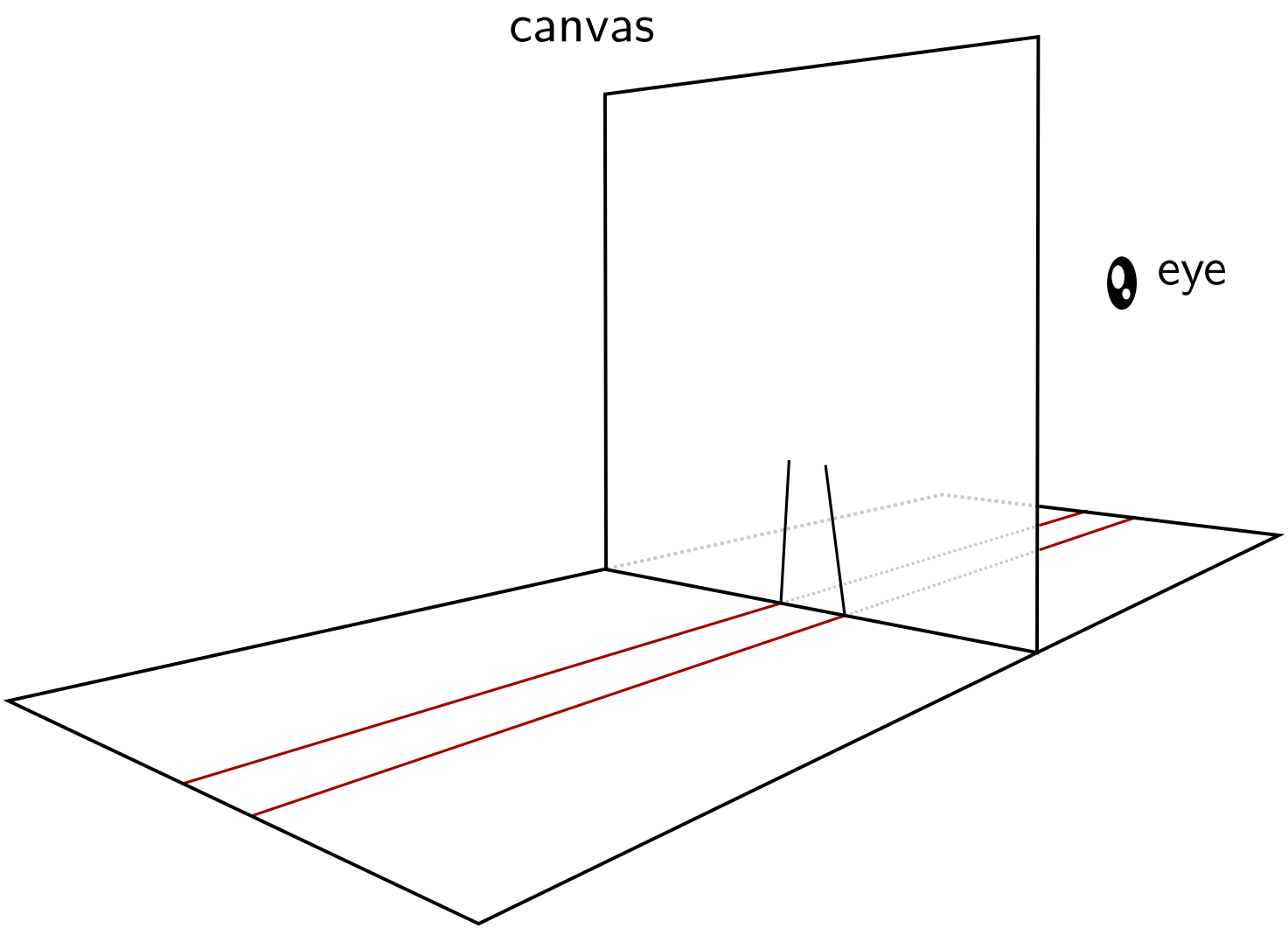

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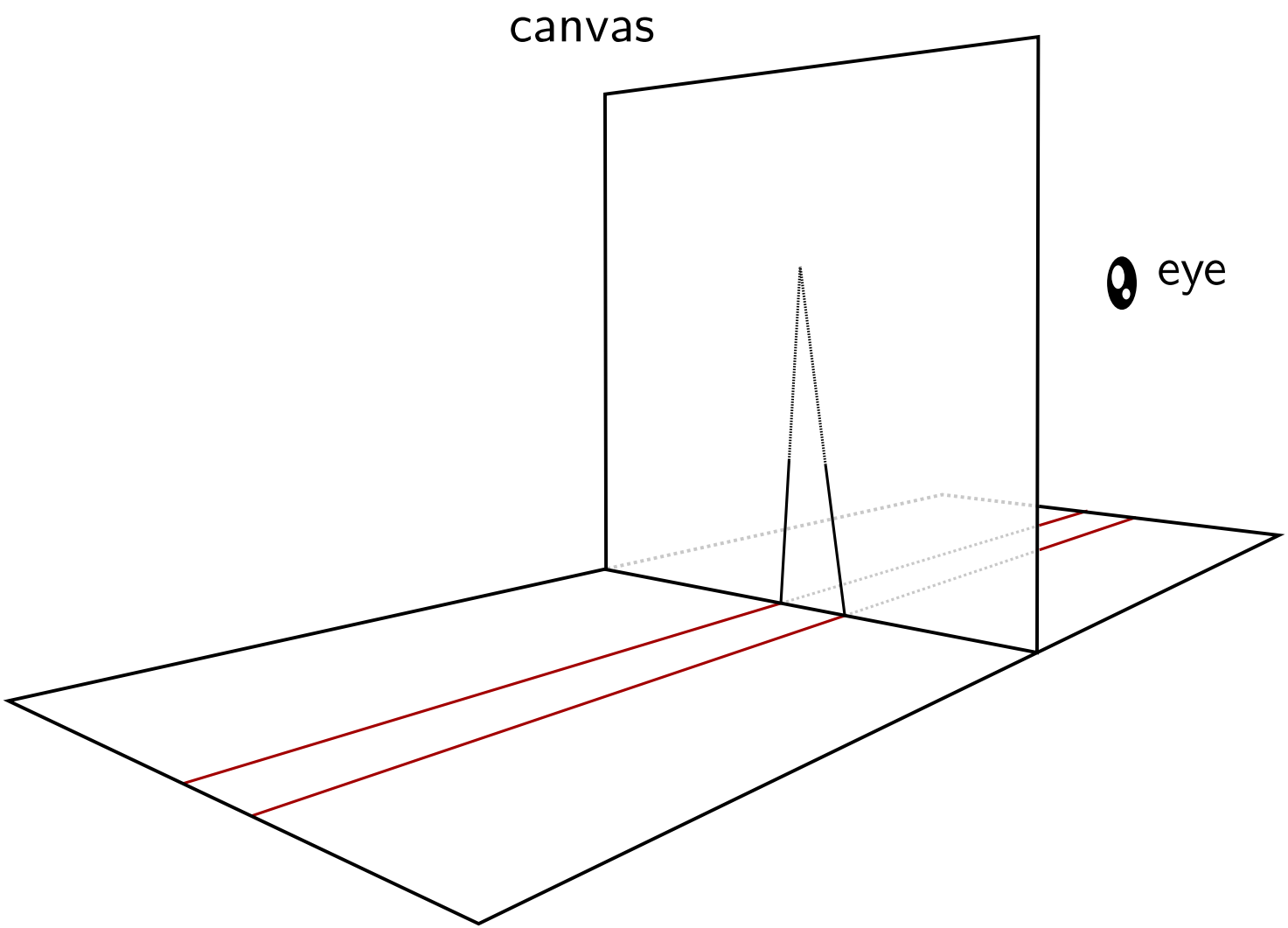
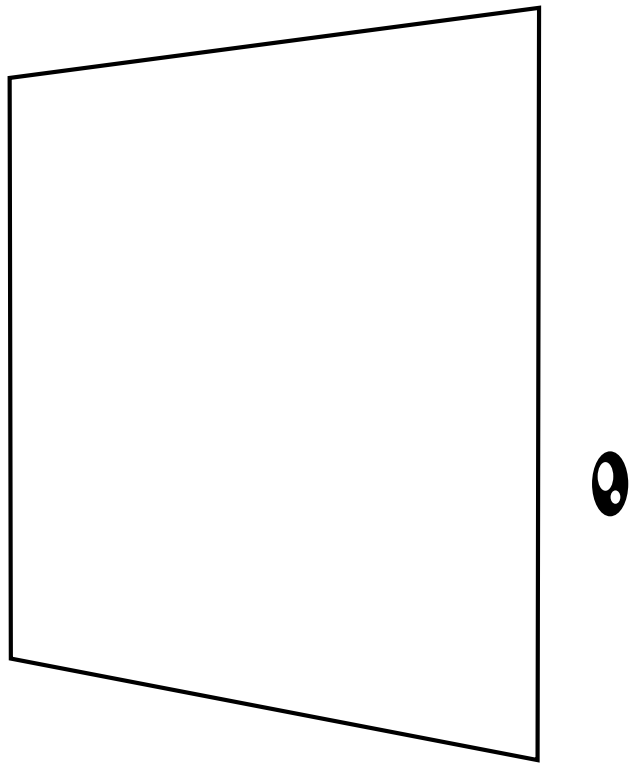
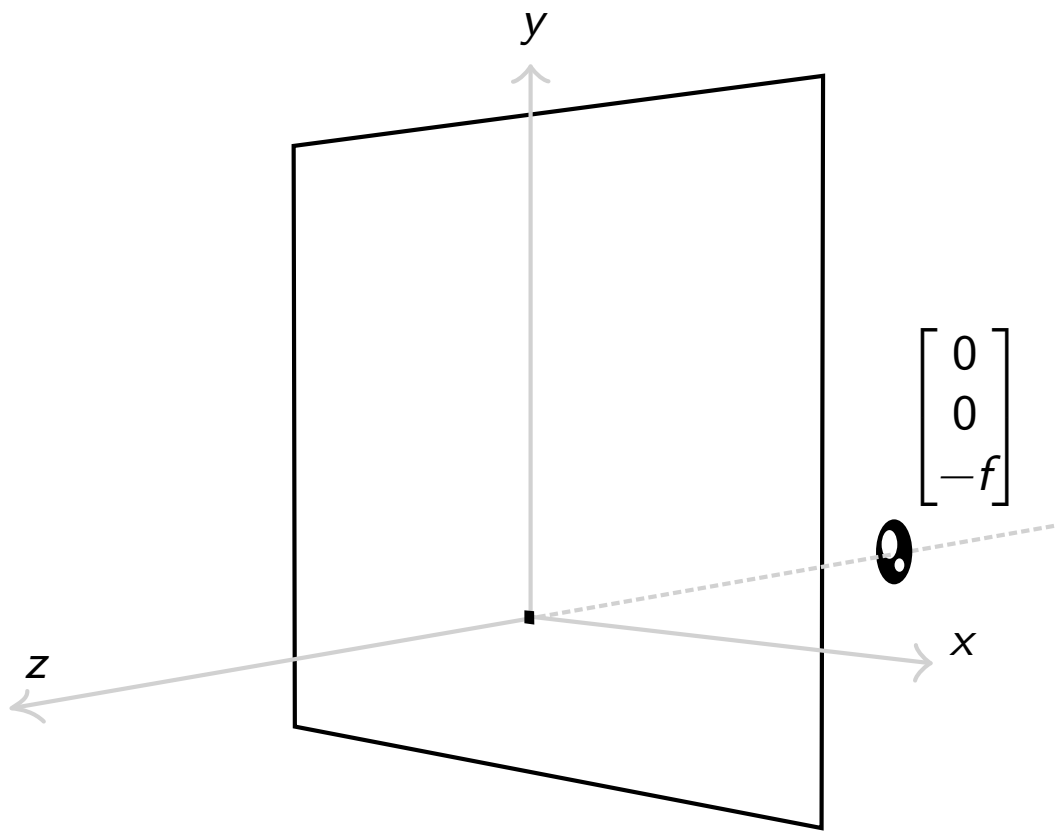


Formula for the Perspective Projection



Formula for the Perspective Projection

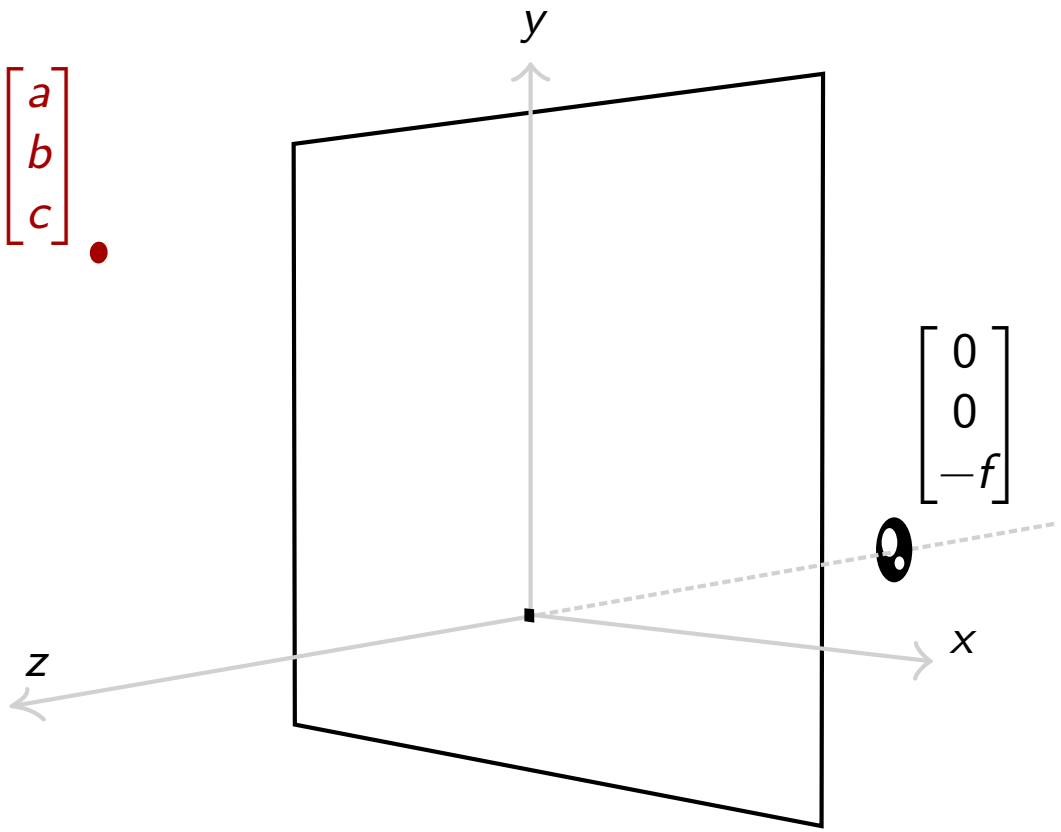
Set up a coordinate system with its xy-plane along the image plane and the eye on the negative z-axis.



Formula for the Perspective Projection

Set up a coordinate system with its xy-plane along the image plane and the eye on the negative z-axis.

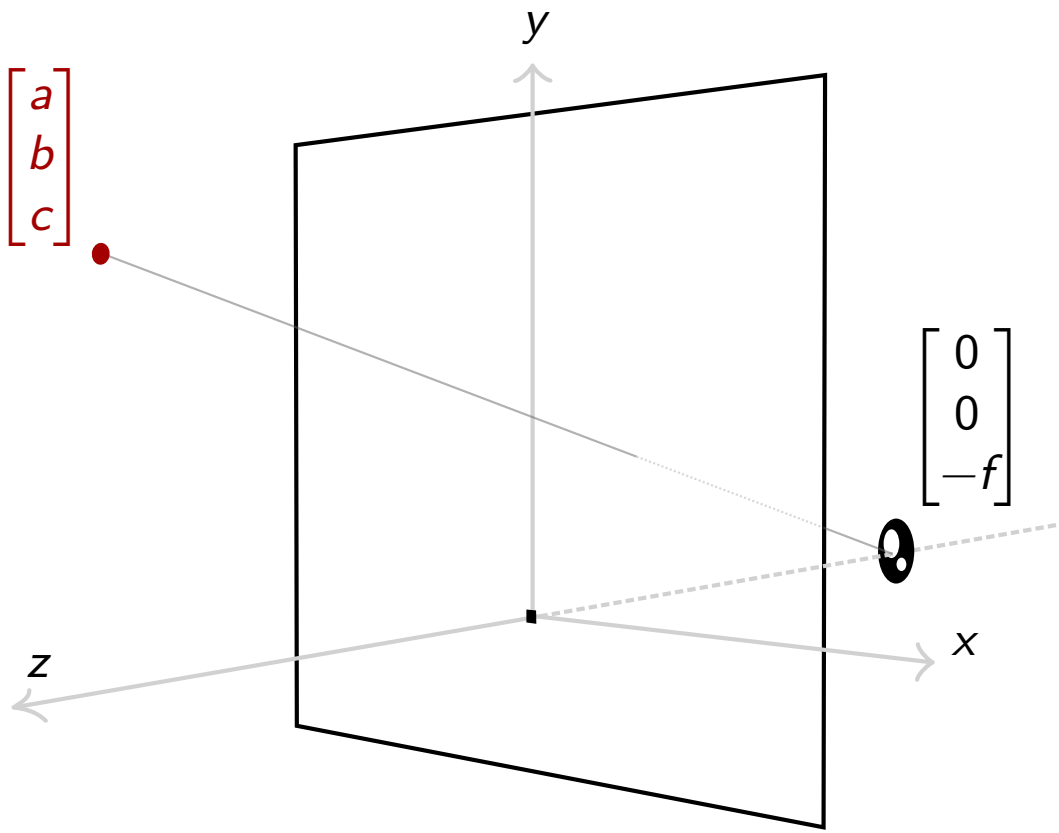
Pick a point (a, b, c) in reality.



Formula for the Perspective Projection

The line from the point to the eye is parametrized by

$$(a, b, c) - t(a, b, c + f).$$



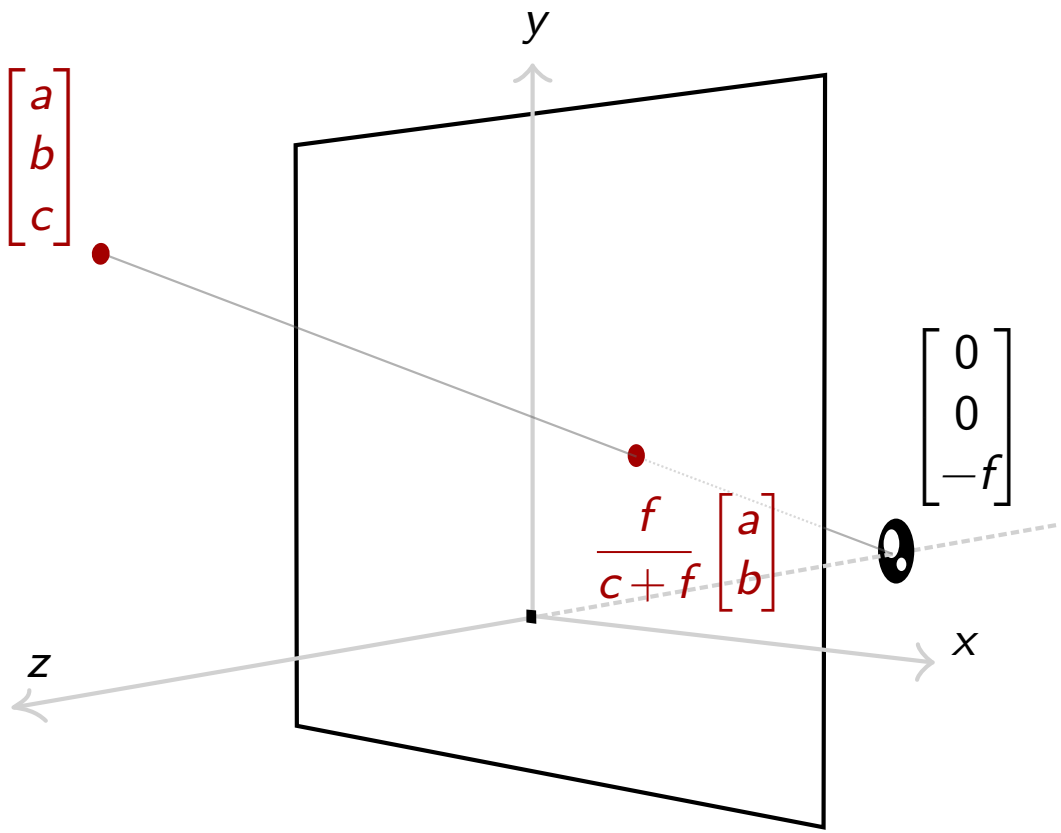
Formula for the Perspective Projection

The line from the point to the eye is parametrized by

$$(a, b, c) - t(a, b, c + f).$$

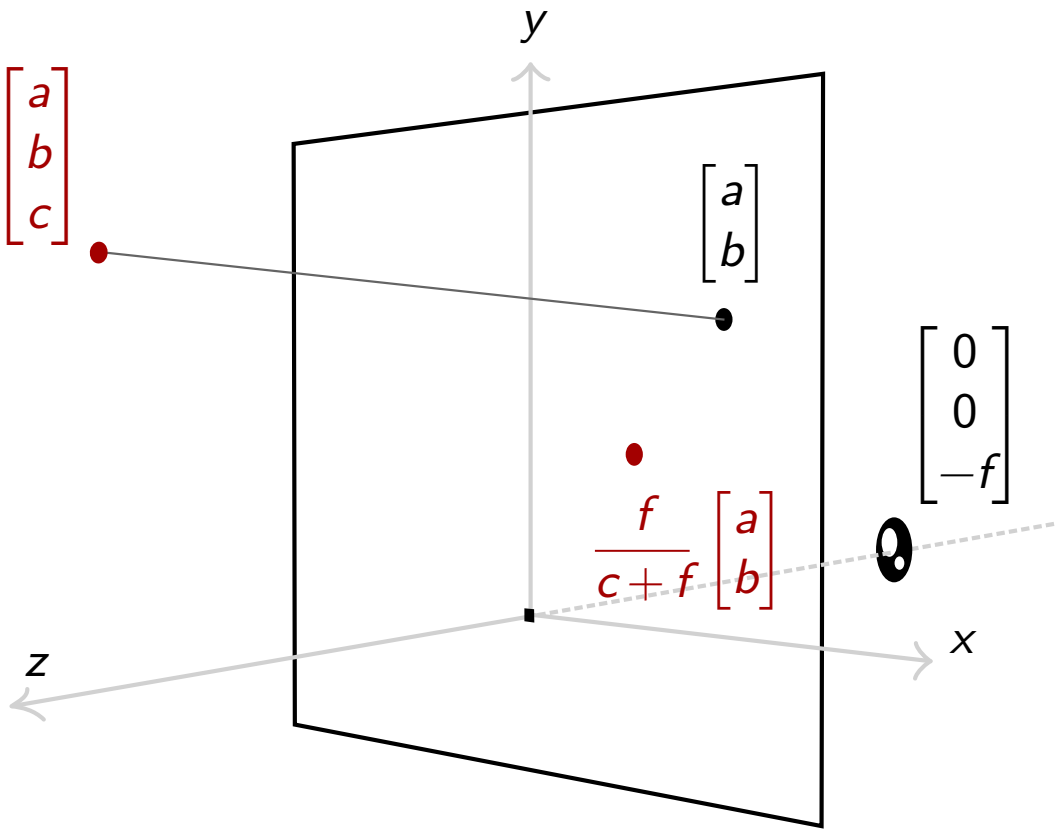
After some calculation, we find that the perspective projection is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \frac{f}{c+f} \begin{bmatrix} a \\ b \end{bmatrix}.$$



Formula for the Perspective Projection

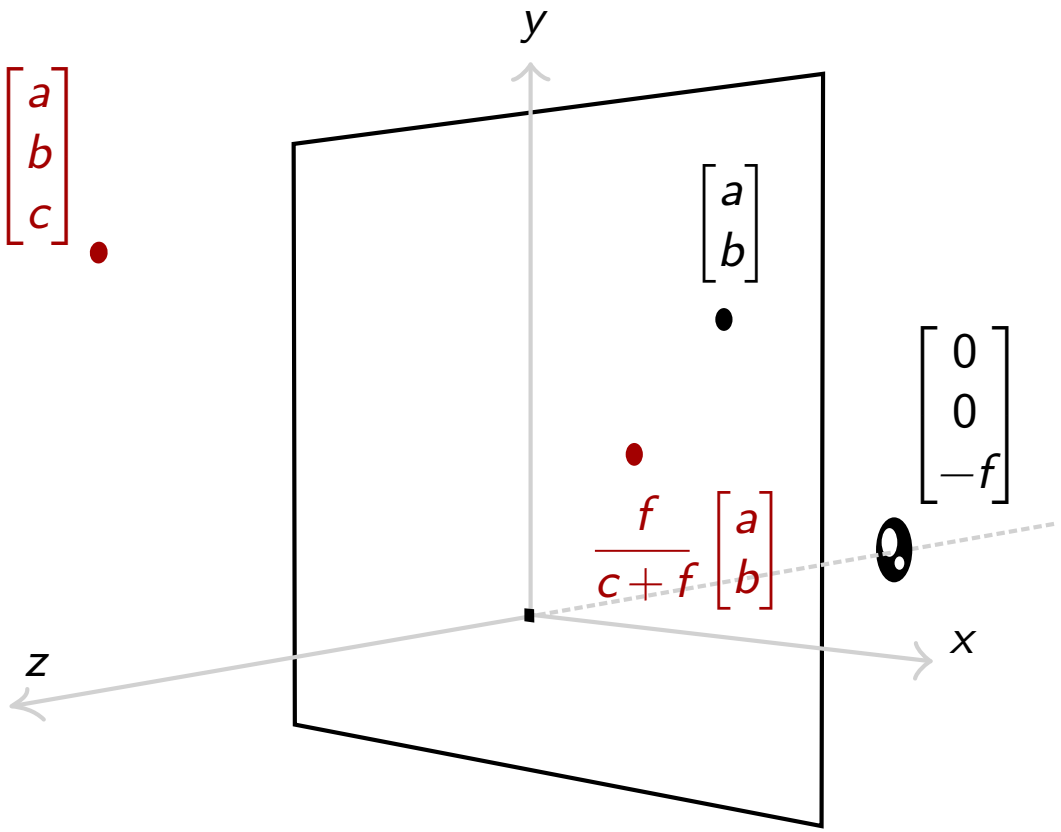
Pushing (a, b, c) back parallel to the z-axis yields the point (a, b) on the image plane.



Formula for the Perspective Projection

Pushing (a, b, c) back parallel to the z-axis yields the point (a, b) on the image plane.

The scalar $f/(c+f)$ is between 0 and 1...



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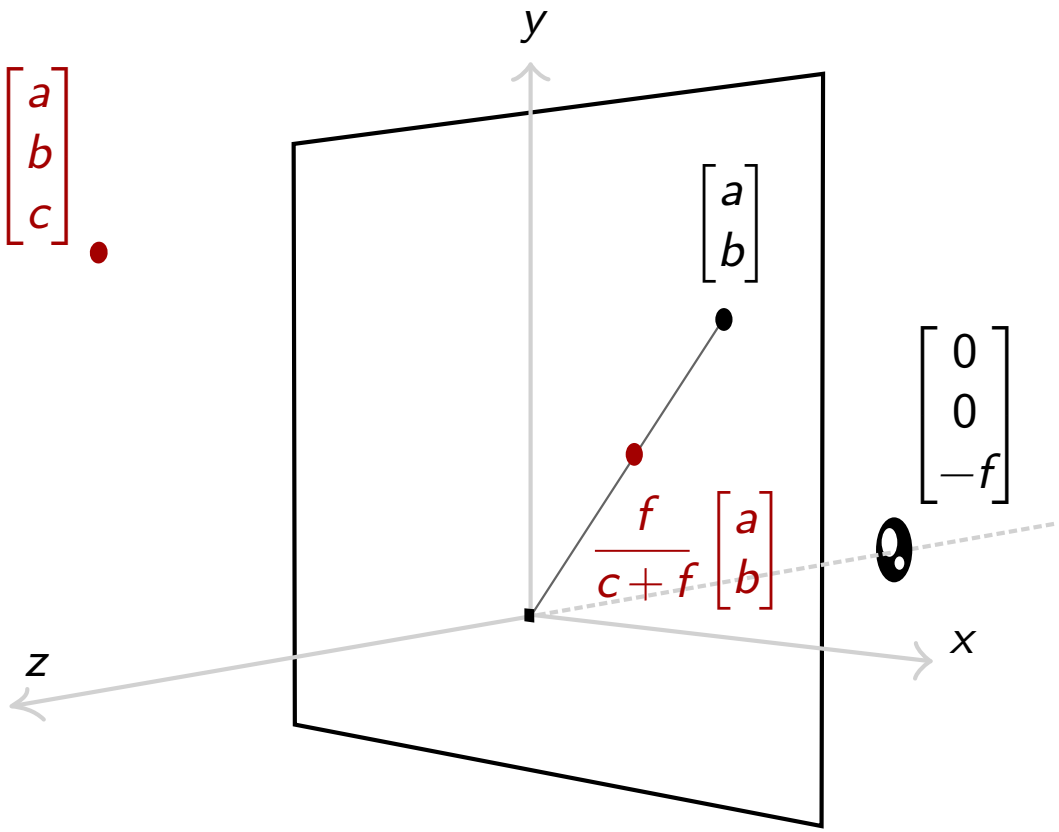
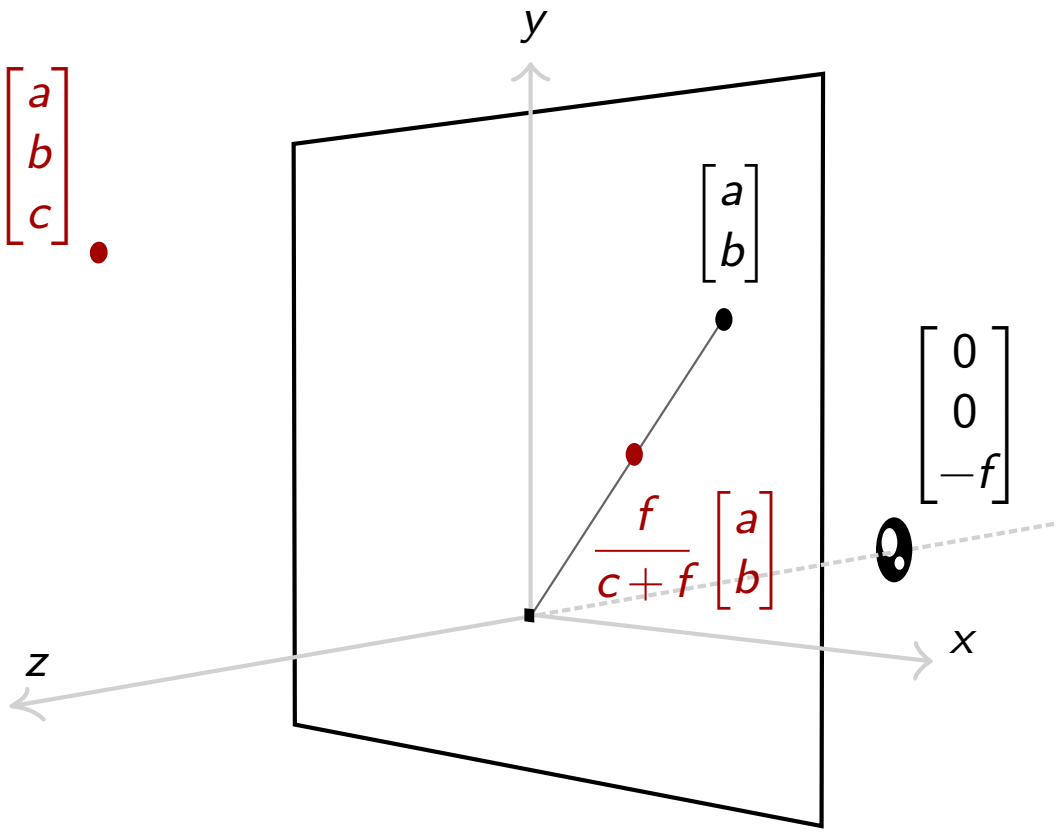
... so multiplying by $f/(c+f)$ pulls points towards the origin.



Formula for the Perspective Projection

Moreover, we have

$$\lim_{c \to \infty} \frac{f}{c+f} = 0.$$

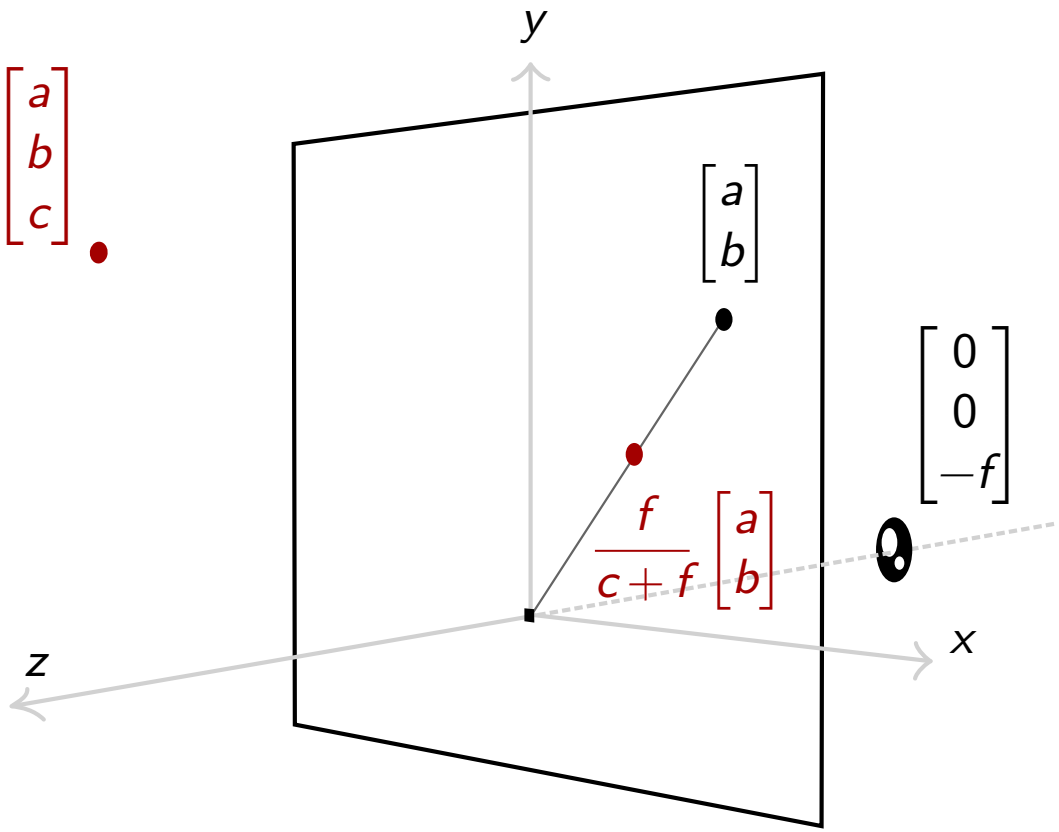


Formula for the Perspective Projection

Moreover, we have

$$\lim_{c \to \infty} \frac{f}{c+f} = 0.$$

So, as (a, b, c) gets further from the canvas, the image point moves closer and closer to the origin!



Railroad Tracks, Again!

Railroad tracks in Lotus, Illinois.

Sight Version

We can also visualize the persepctive projection as a model of sight.

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This model makes two important simplifications:

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1. We treat the pupil as a single point (it's actually a little lens).

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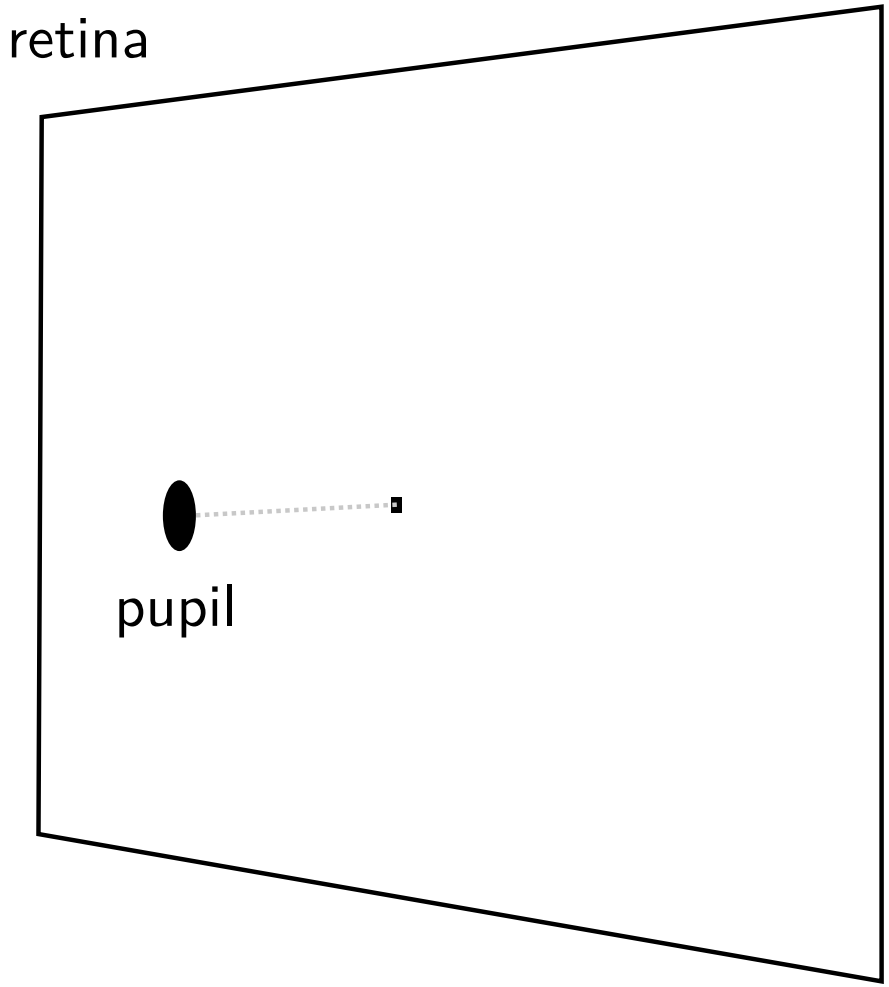
We can also visualize the persepctive projection as a model of sight.

This model makes two important simplifications:

1. We treat the pupil as a single point (it's actually a little lens).
2. We treat the retina as a plane (it's actually rounded).

Sight Version



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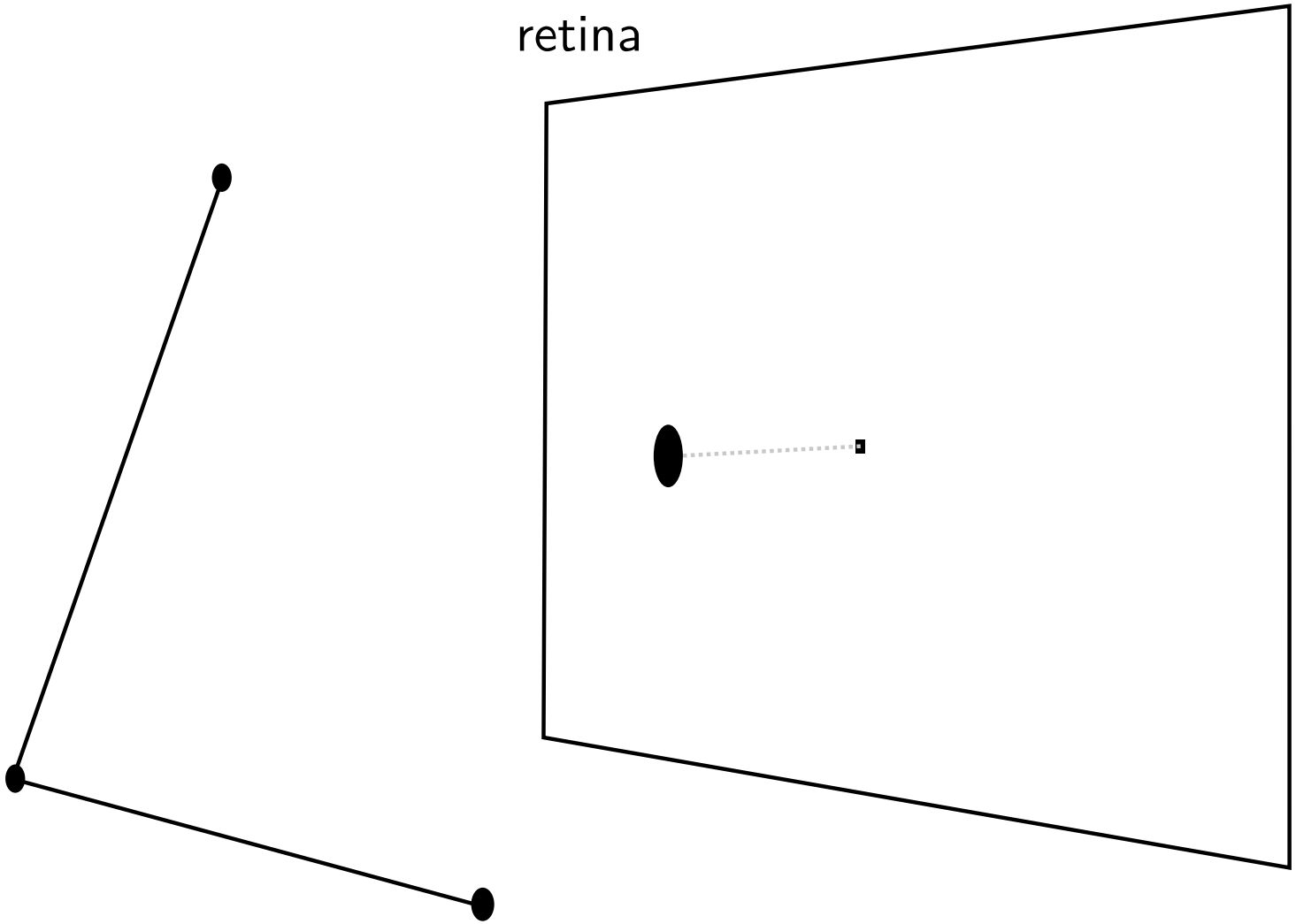

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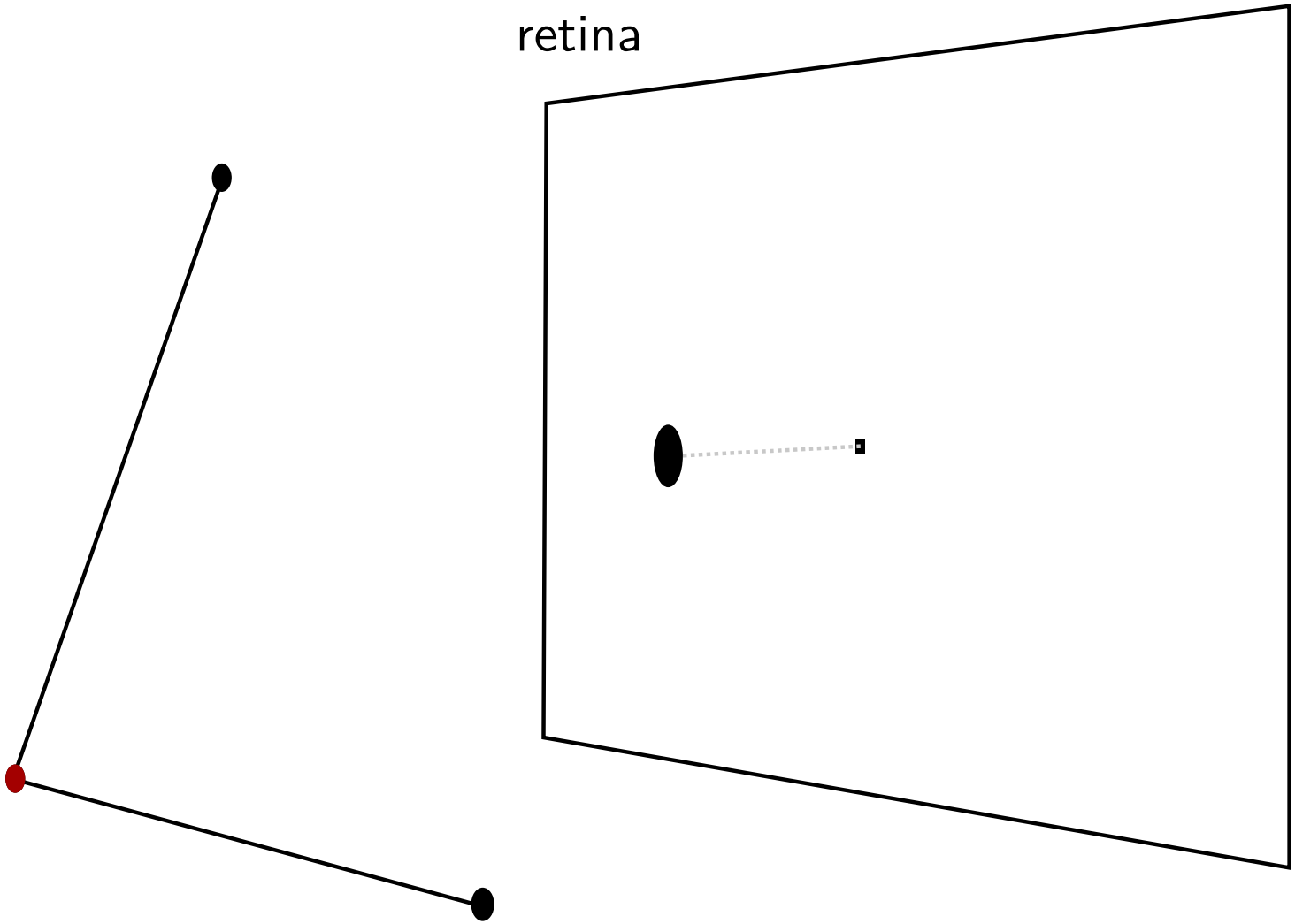

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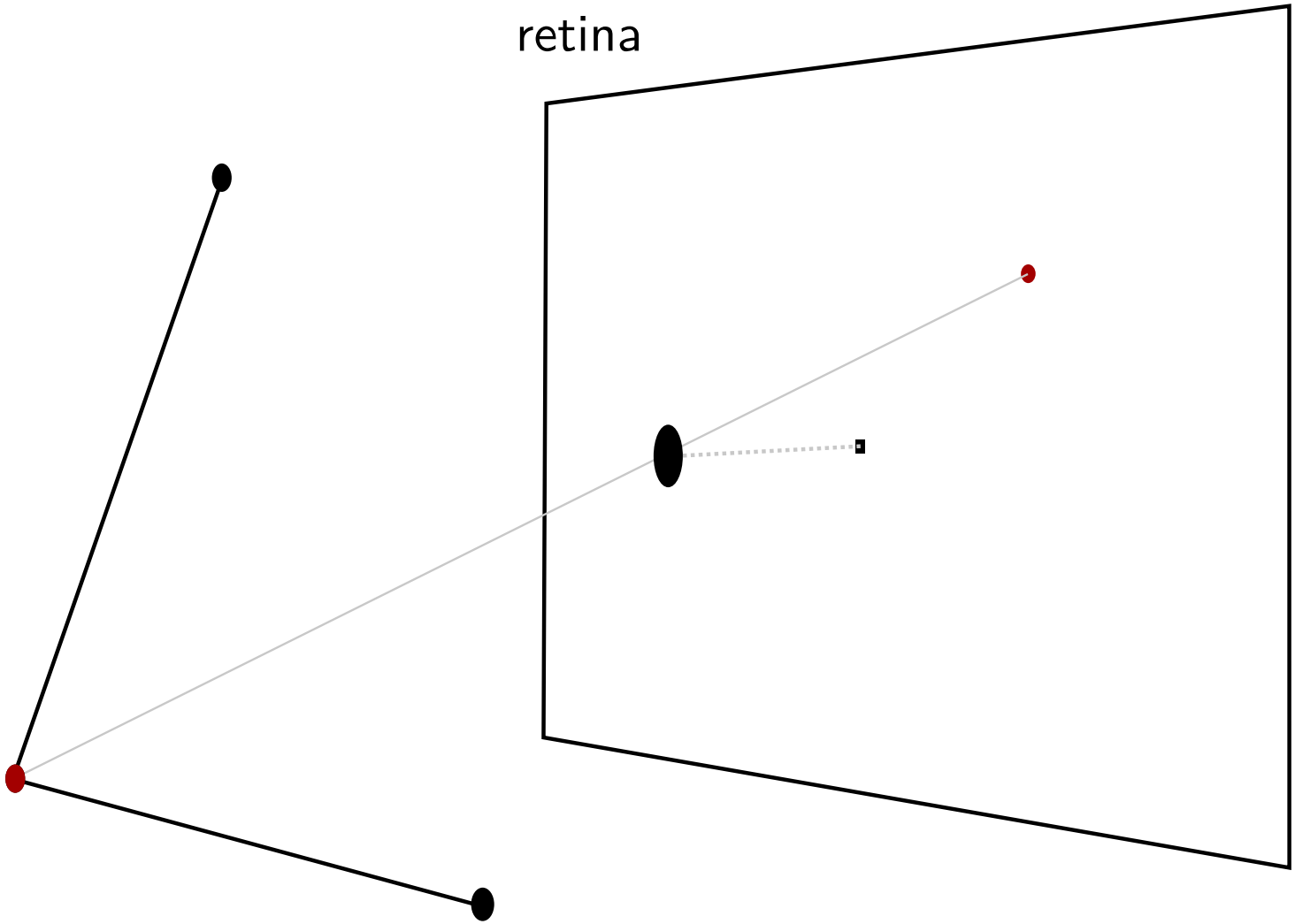

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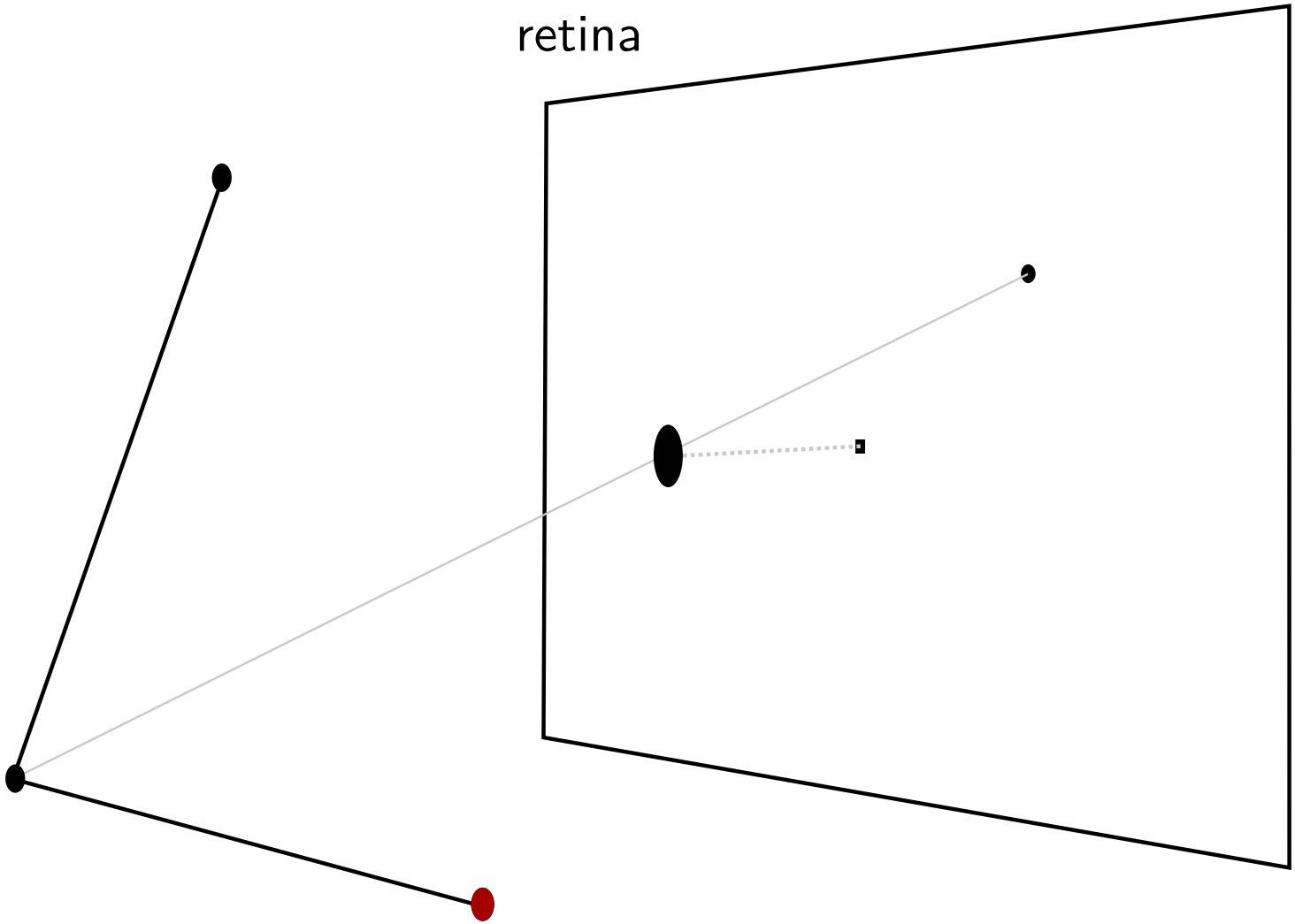

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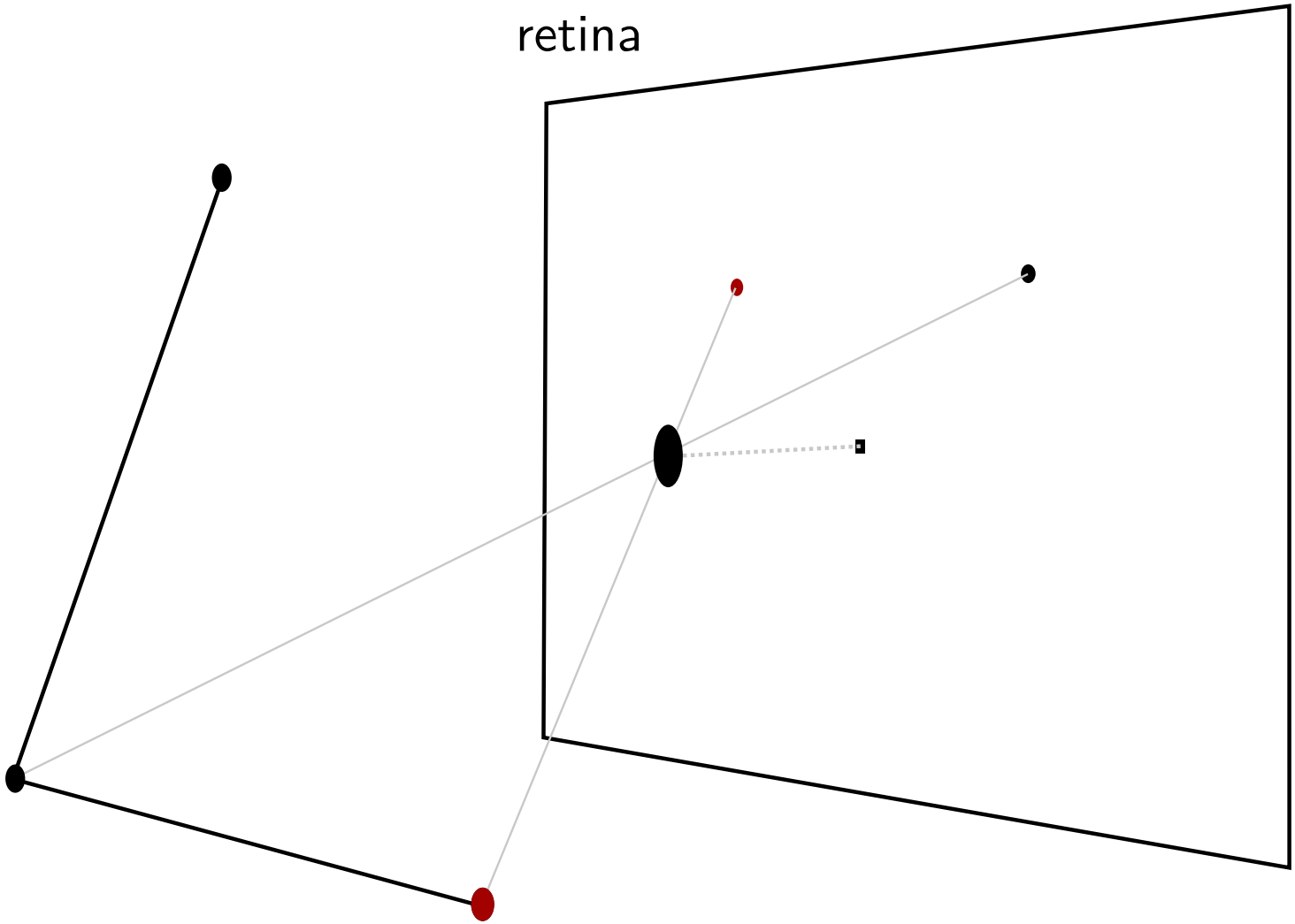

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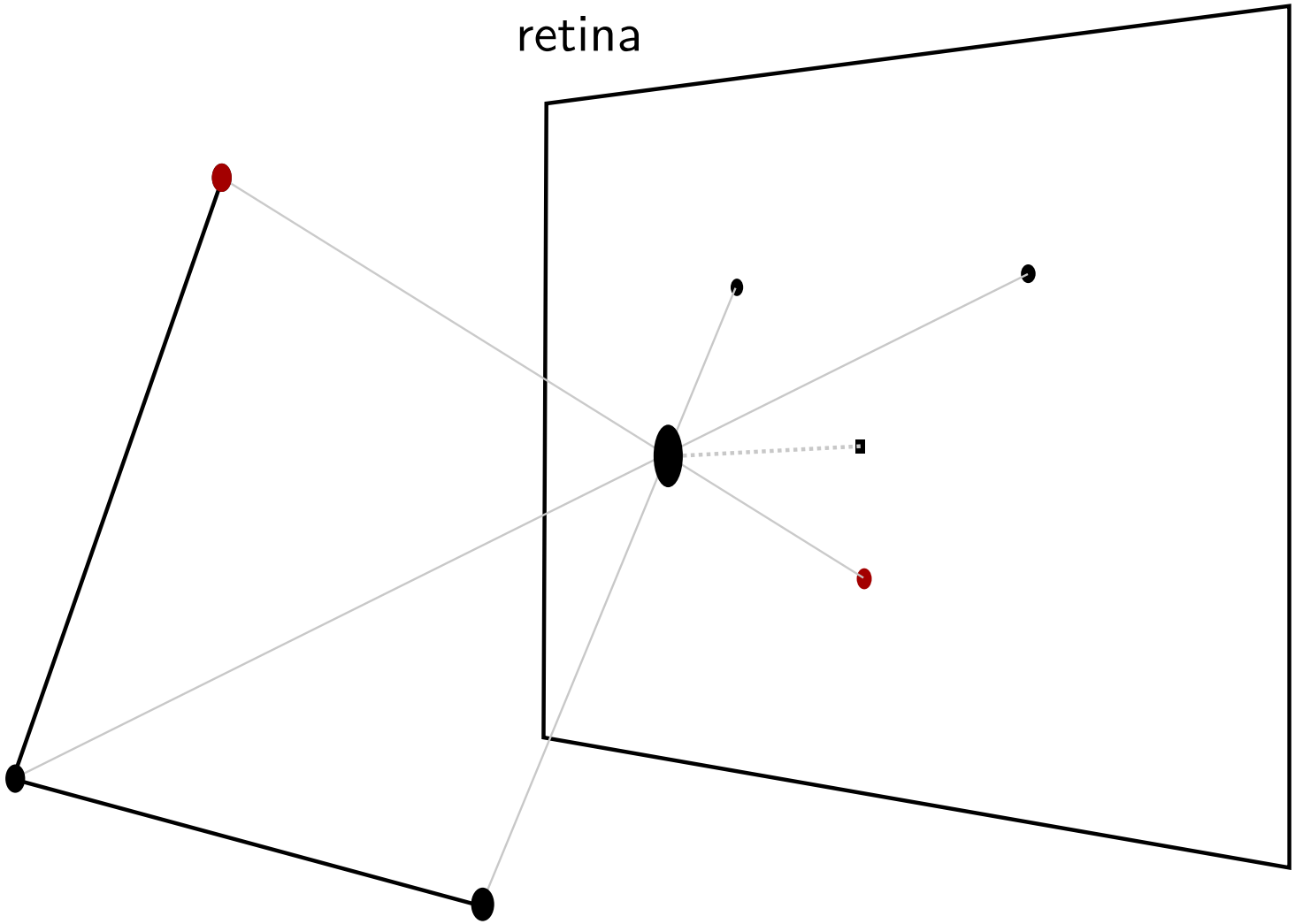

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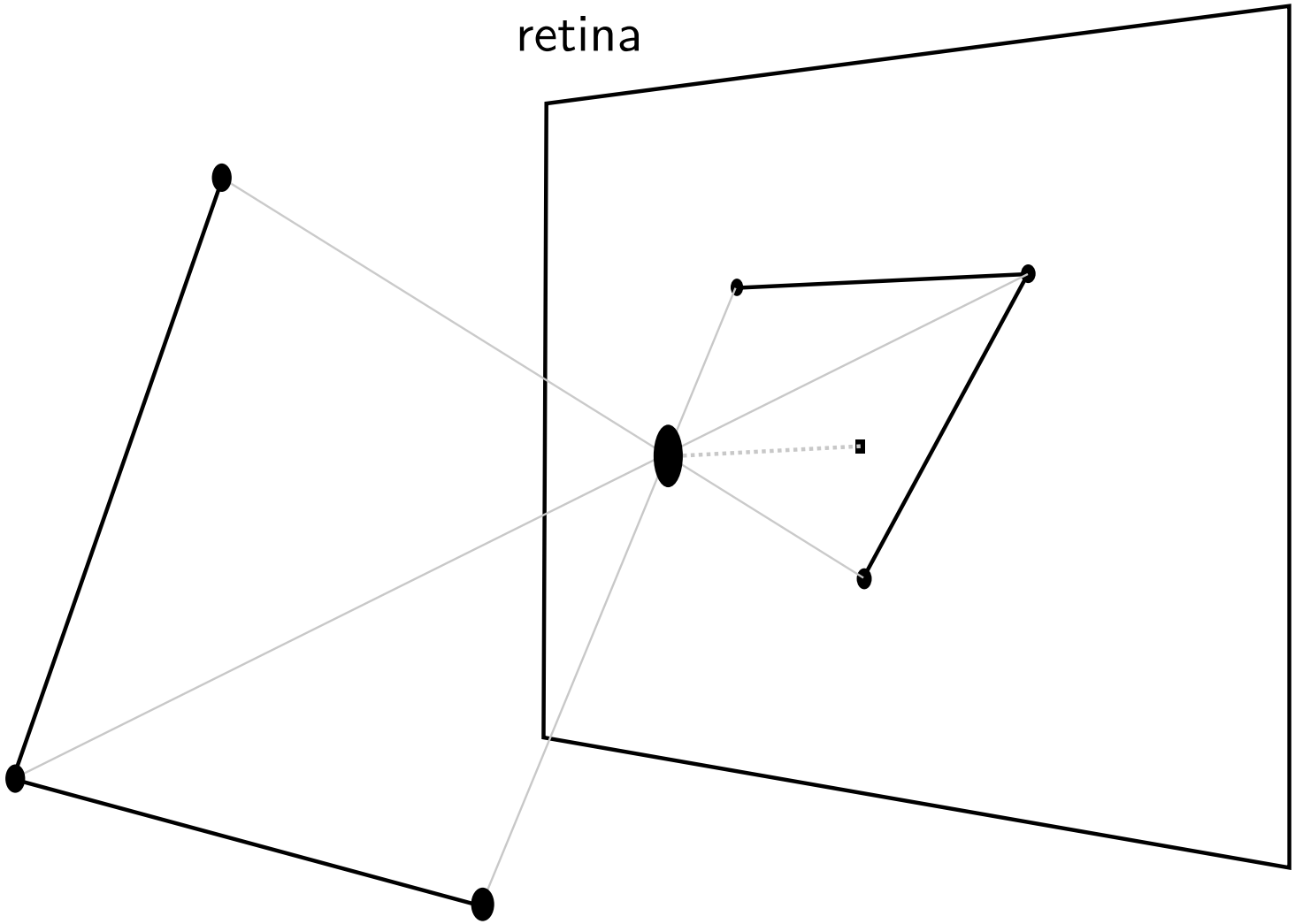

Sight Version

This is identical to the drawing version, except now the "center of projection" (ie, the pupil) is *in front of* the image plane (ie, on the *positive* z-axis).

Sight Version

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The images of an object under the two perspective projections centered at $(0, 0, f)$ and $(0, 0, -f)$ are exactly the same, except that one is reflected upside-down and left-to-right compared to the other.

European Classical Era

There's some evidence that Ancient Romans intuited linear perspective.

Ancient Roman Example

Wall painting from the Villa of P. Fannius Synistor at Boscoreale, near Pompeii (before 76 CE).

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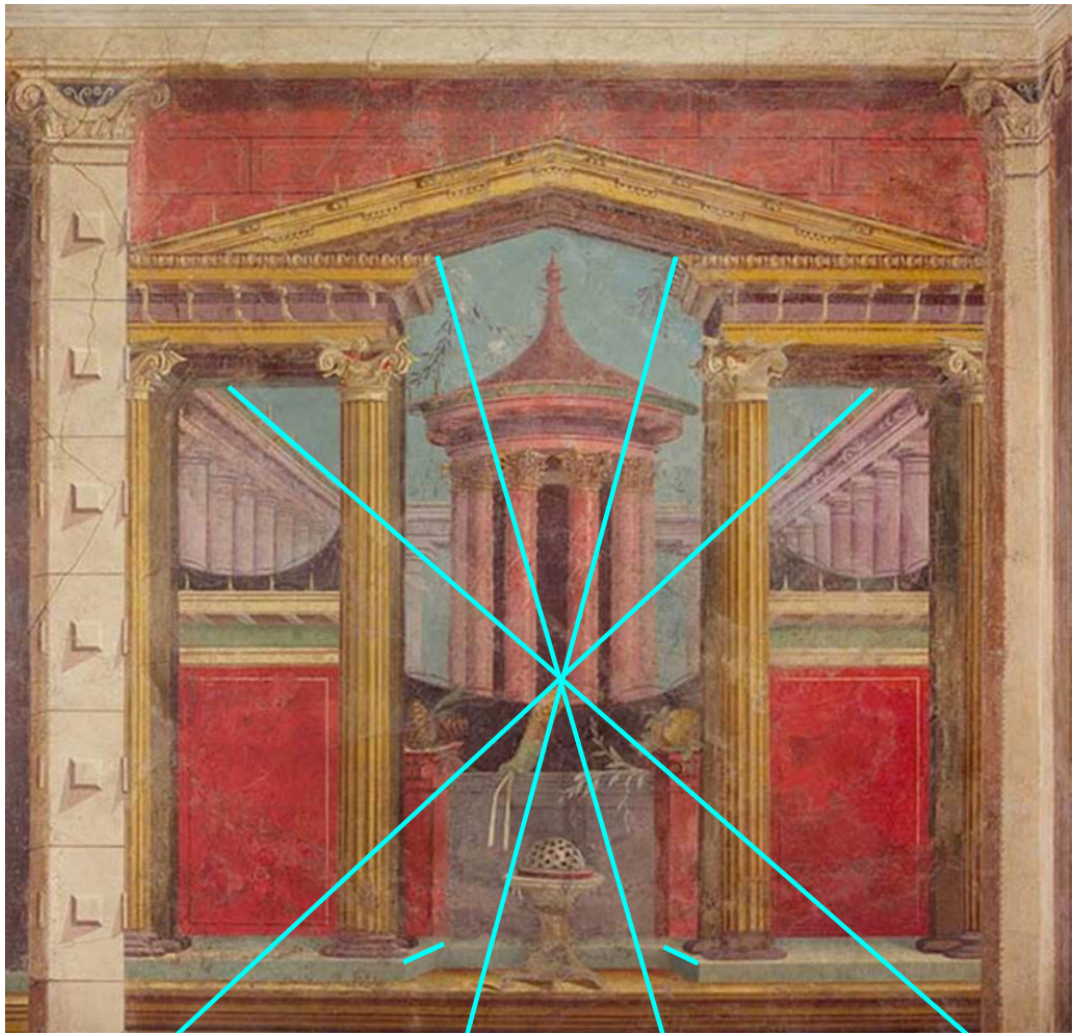

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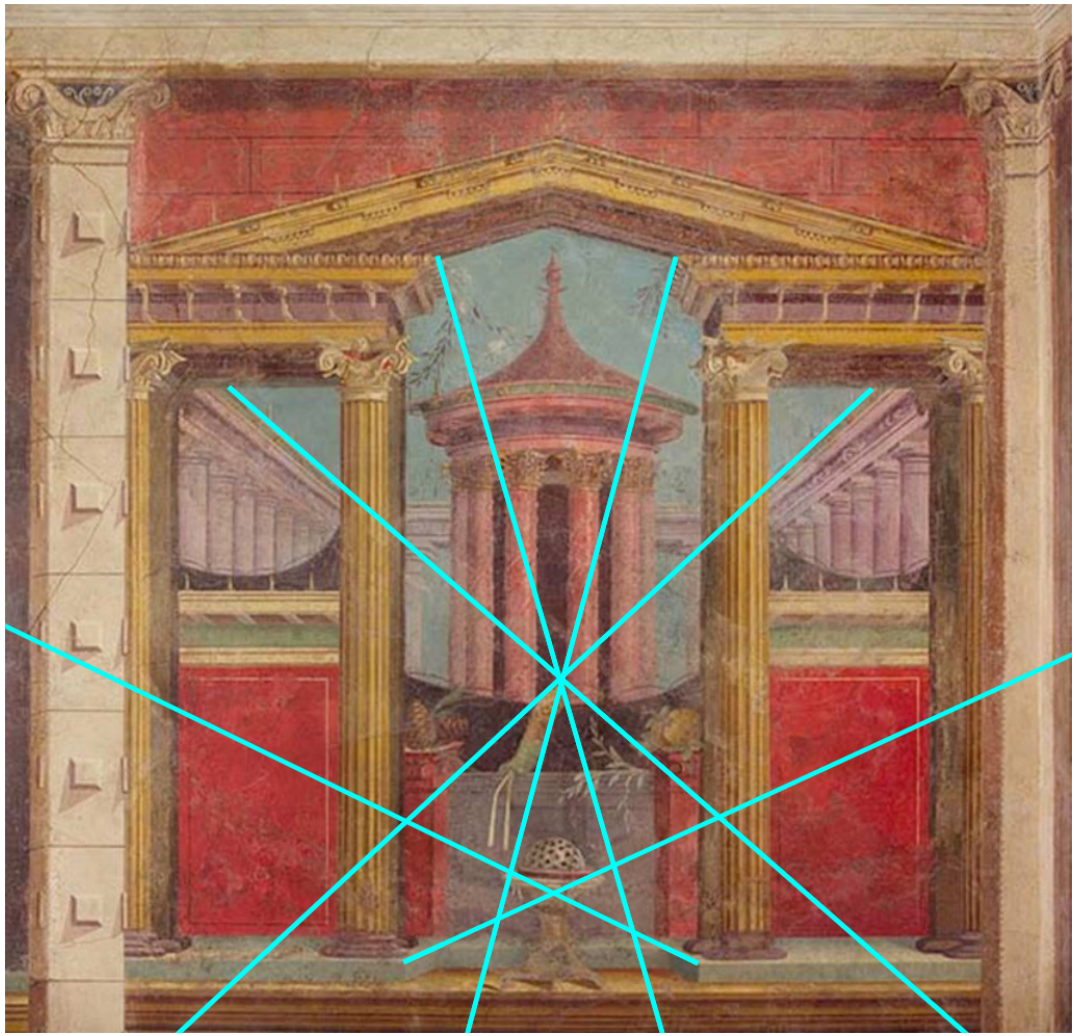

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European Classical Era

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There's not much evidence that Ancient Greeks or Roman painters had a clear underlying mathematical model of linear perspective.

European Medieval Era

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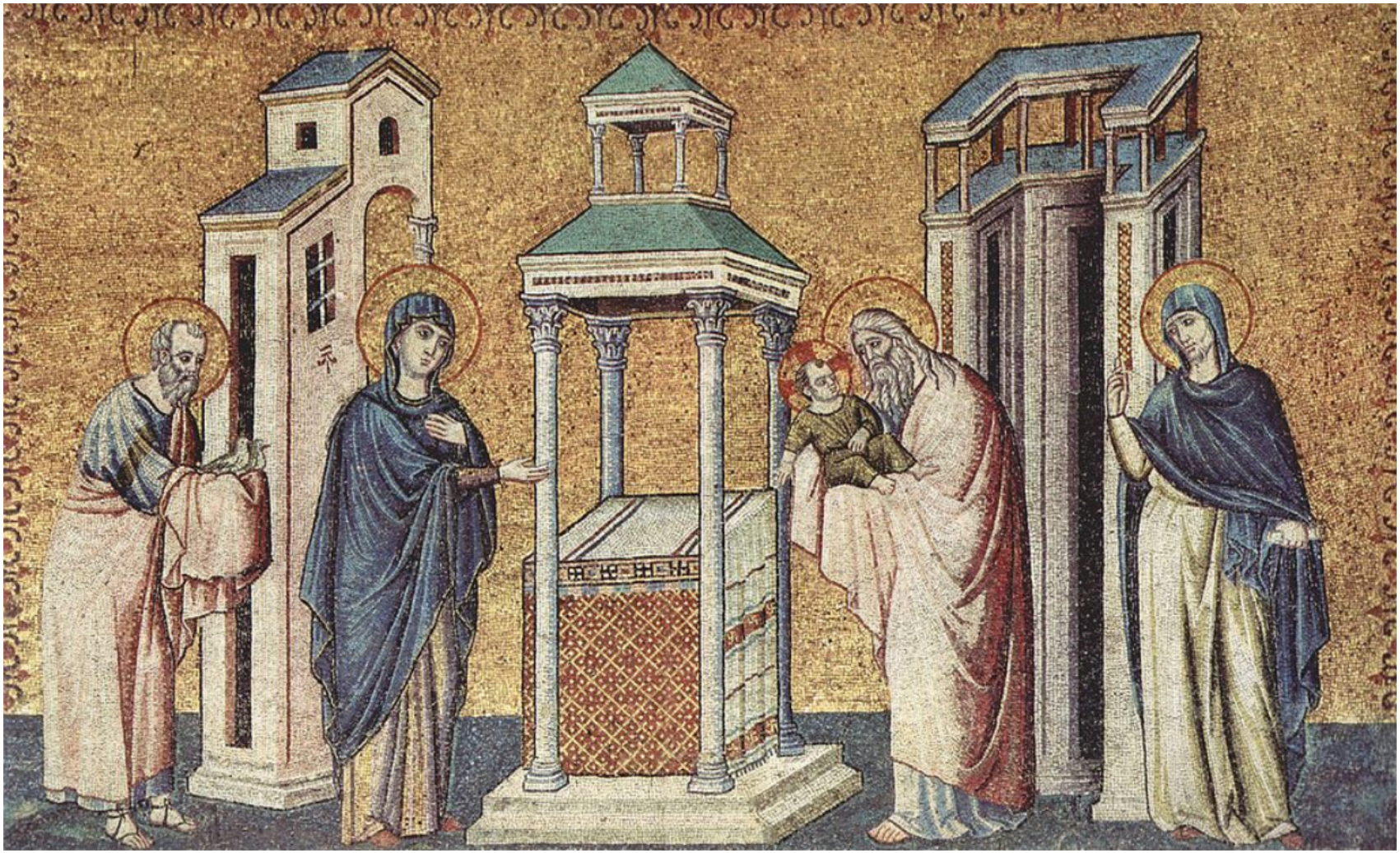
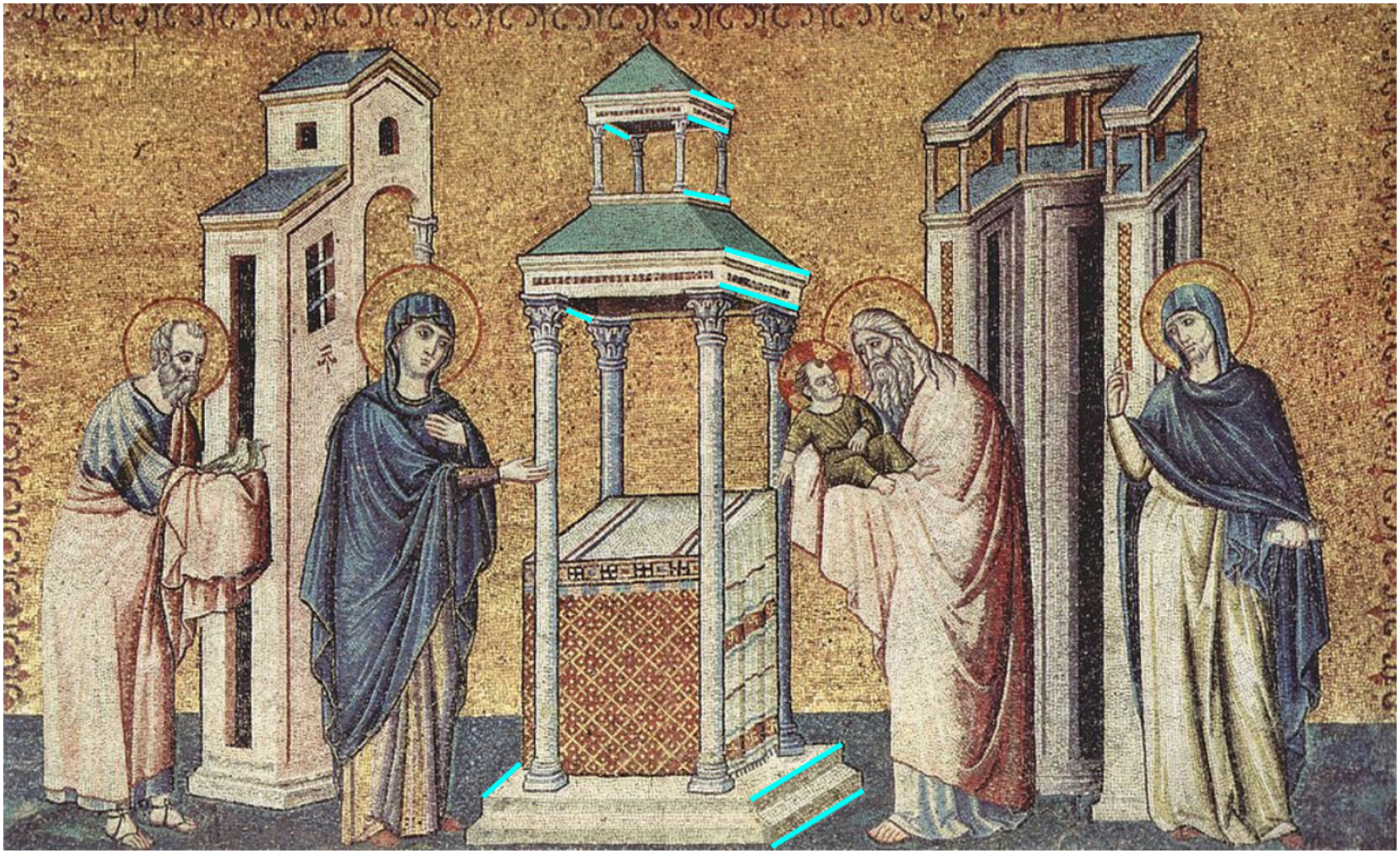
Intuition about linear perspective does not persist very well into the medieval era. Depictions of depth become very ad hoc.

Medieval Perspective Fail



Mosaic (c. 1291) depicting the presentation of Jesus at the temple, by Pietro Cavallini.

Medieval Perspective Fail



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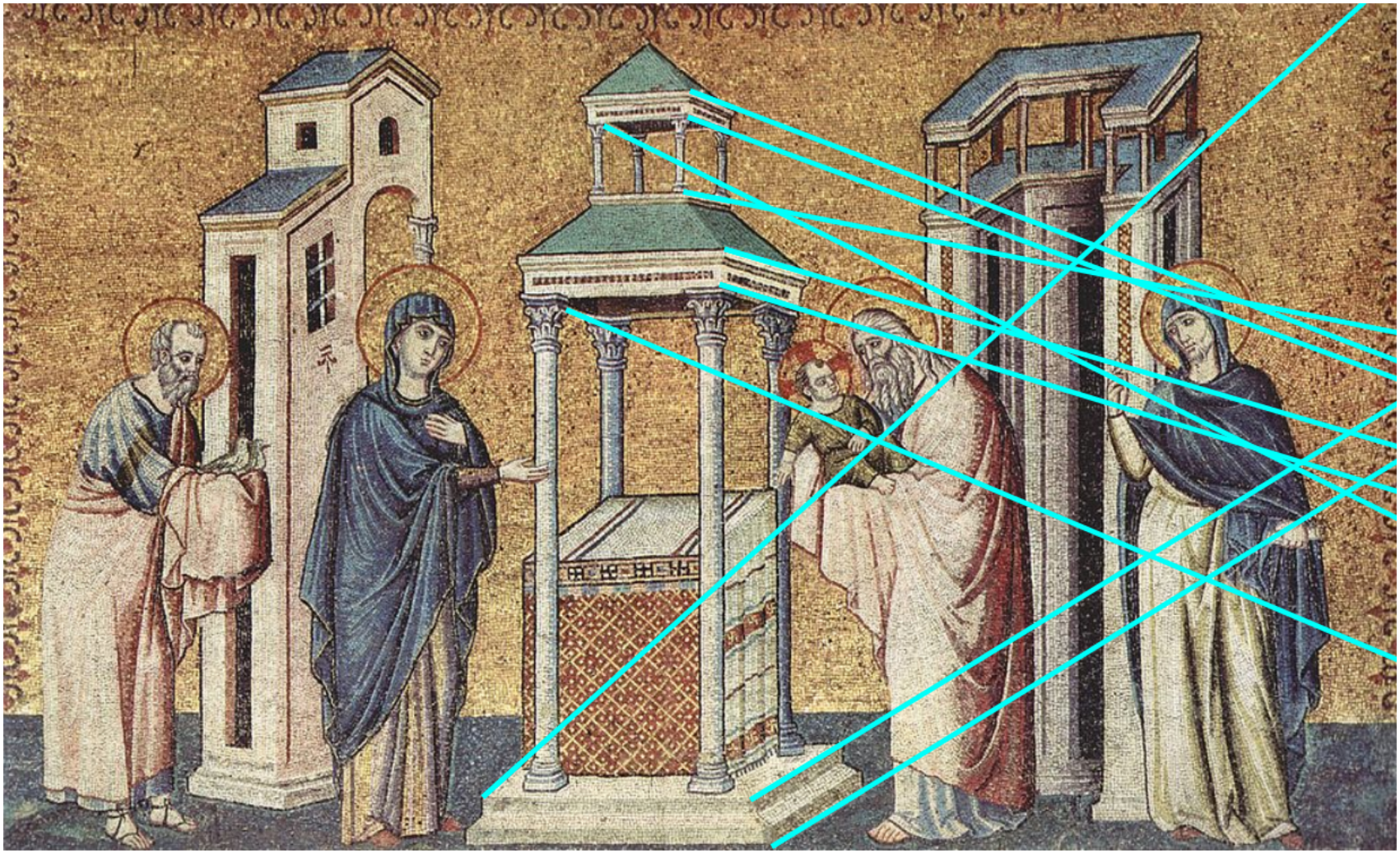

Mosaic (c. 1291) depicting the presentation of Jesus at the temple, by Pietro Cavallini.

Middle Eastern Golden Age

During Europe's medieval era, Middle Eastern scholars engaged critically with European classical era thought.

Manuscript from the 1200s depicting Socrates.

Classical Era Speculation

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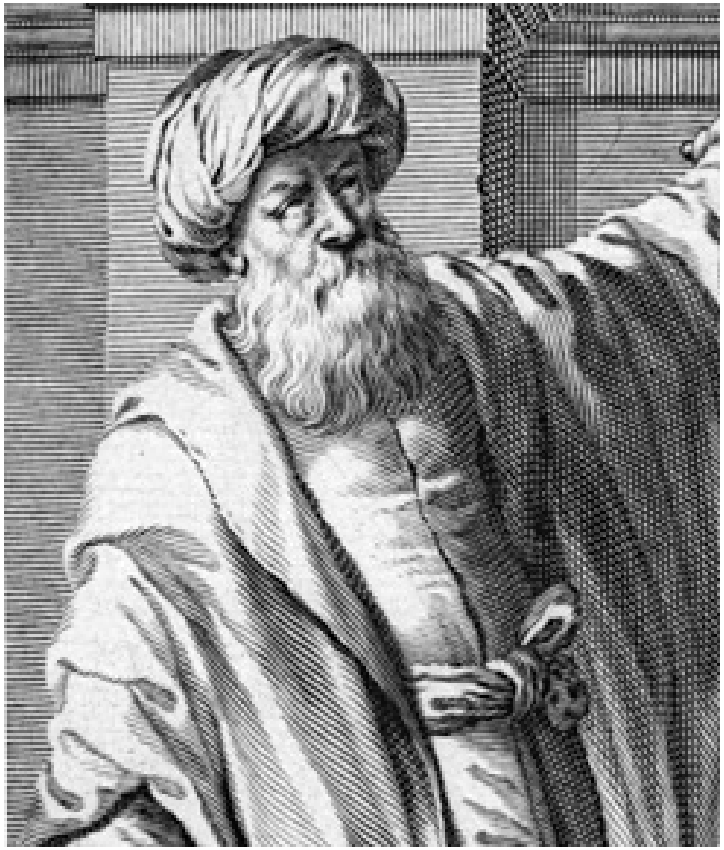
But both sides were sort of just speculating...

Alhazen



From Johannes Hevelius's *Selenographia* (1647).

... until Alhazen (ابن الهيثم, c. 965–1040) came in with experimental data.

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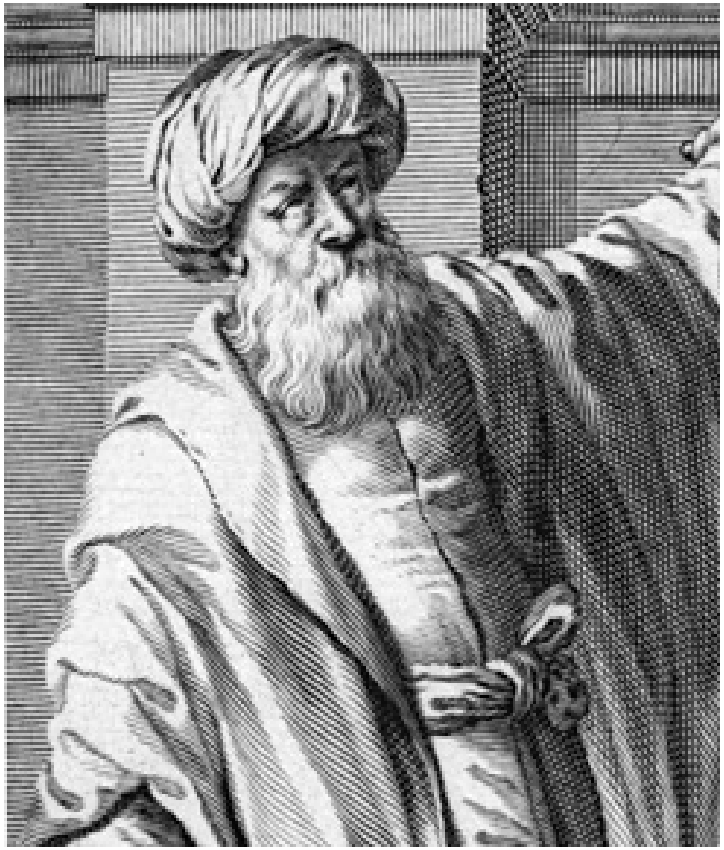

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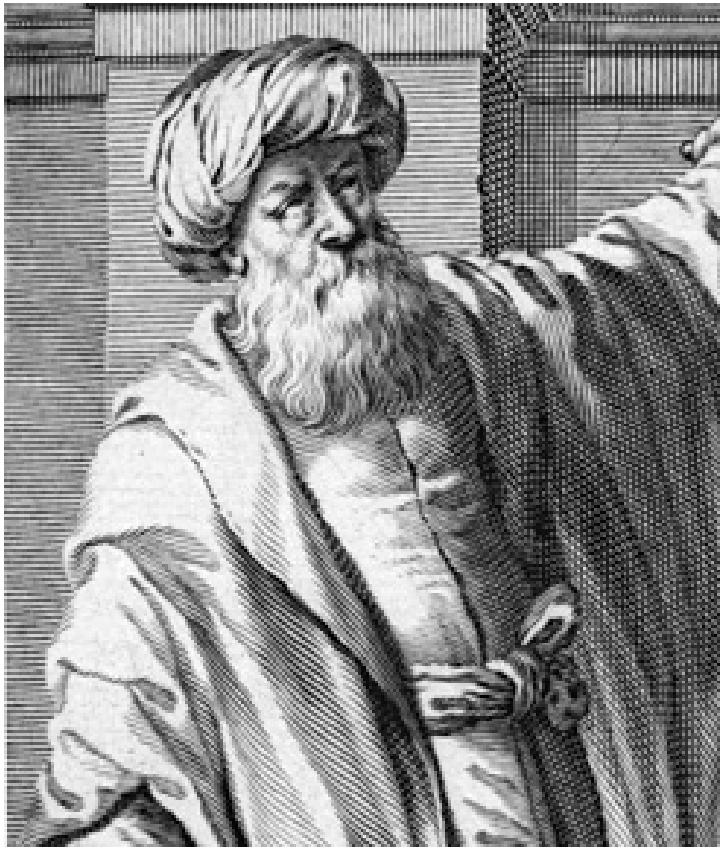

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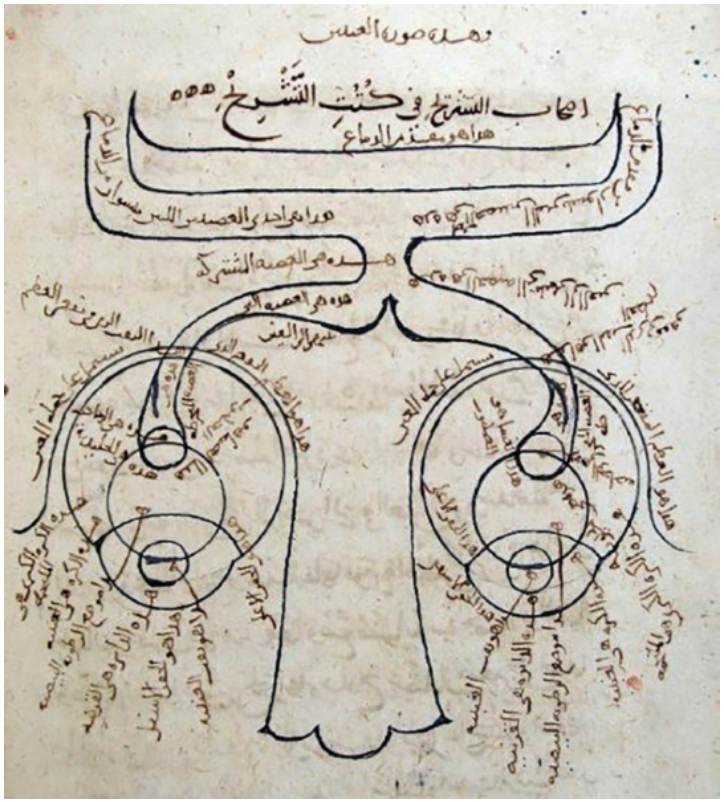
He argued against the extramission theory and proposed what is *almost* the modern theory of sight: light reflects off of objects and enters our eyes.

Alhazen



Anatomy of the human eye from Alhazen's *Book of Optics*.

He understood that the image on the retina would have to be inverted...

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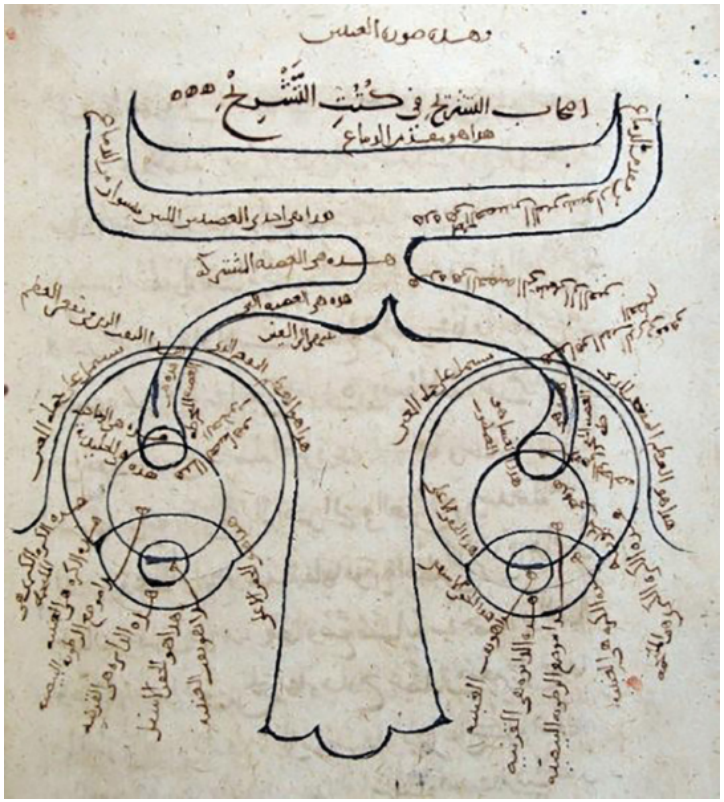

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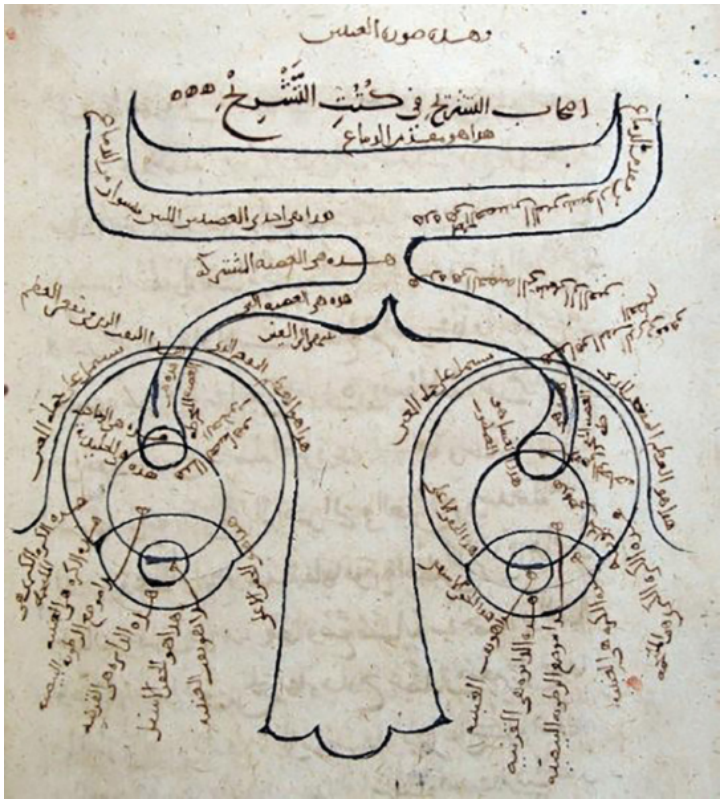

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He understood that the image on the retina would have to be inverted...

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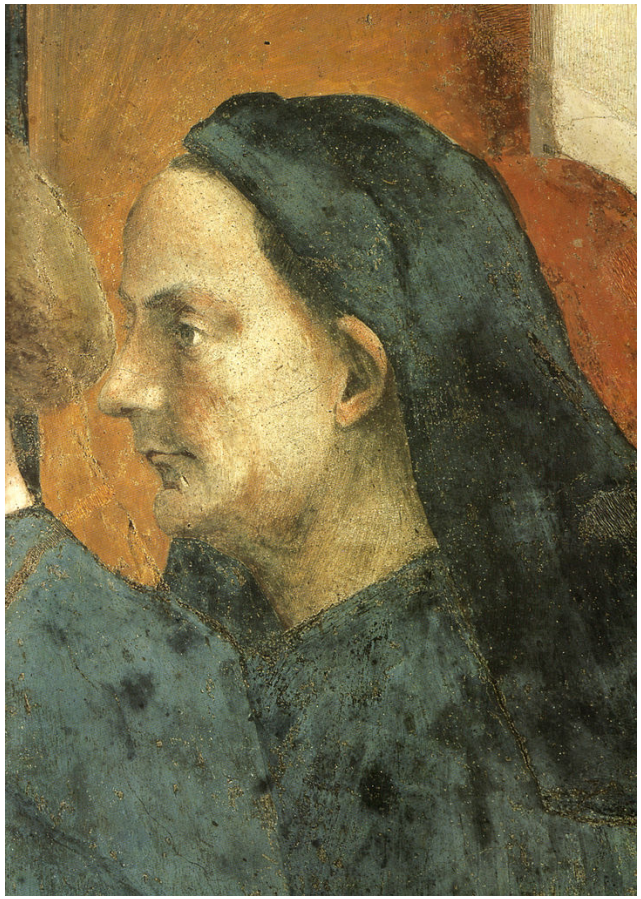
The *Book of Optics* was translated into Latin around 1200, and was very influential in Europe.

Renaissance Italy

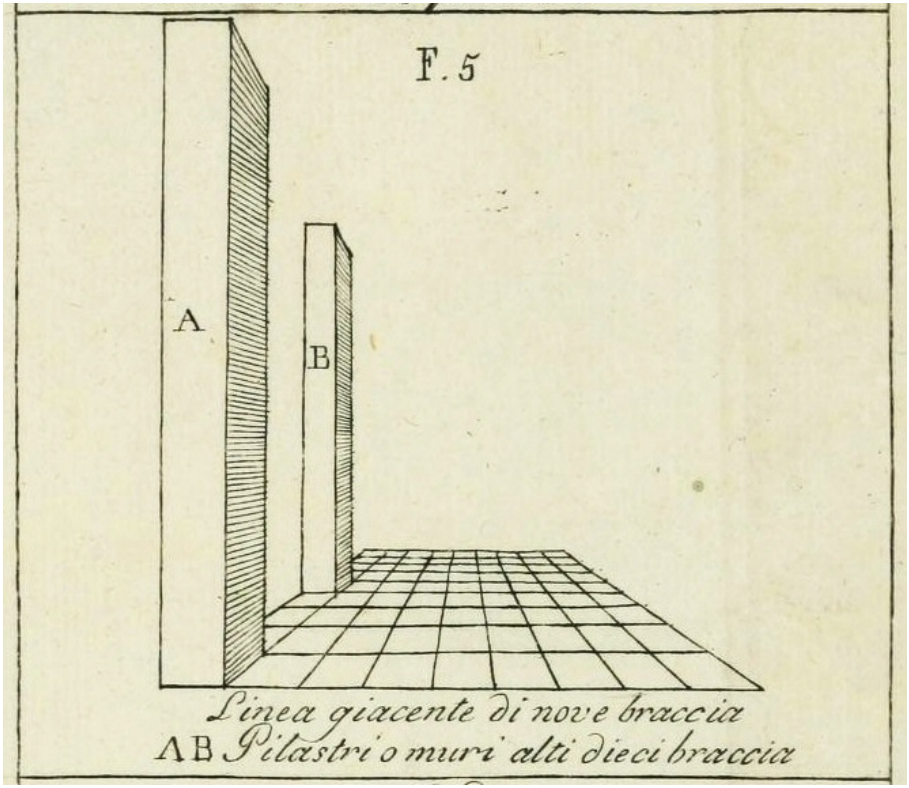
The architect Filippo Brunelleschi (1377–1446) formalized the idea of linear perspective around 1415.



Portrait of Brunelleschi (c. 1425) by Masaccio.

Renaissance Italy

The first written account of linear perspective occurs in *De Pictura* (1435) by Brunelleschi's friend, Leon Battista Alberti (1404–1472).



From an 1804 edition of Alberti's *Della Pictura*.

Renaissance Italy

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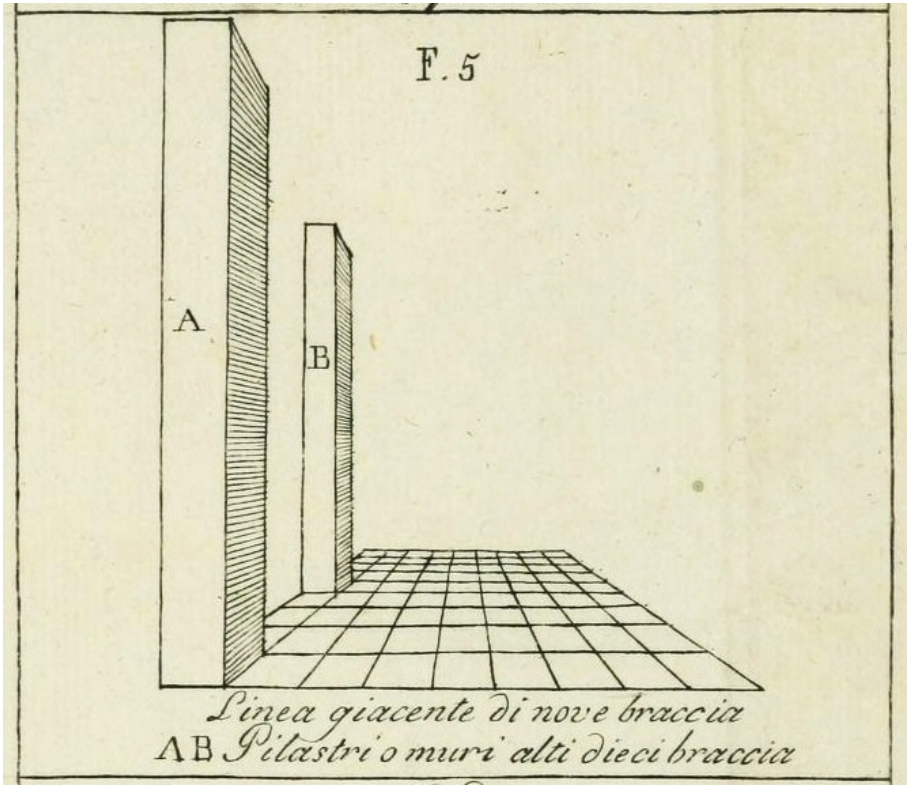
Both Brunelleschi and Alberti knew about Alhazen's theory of vision.



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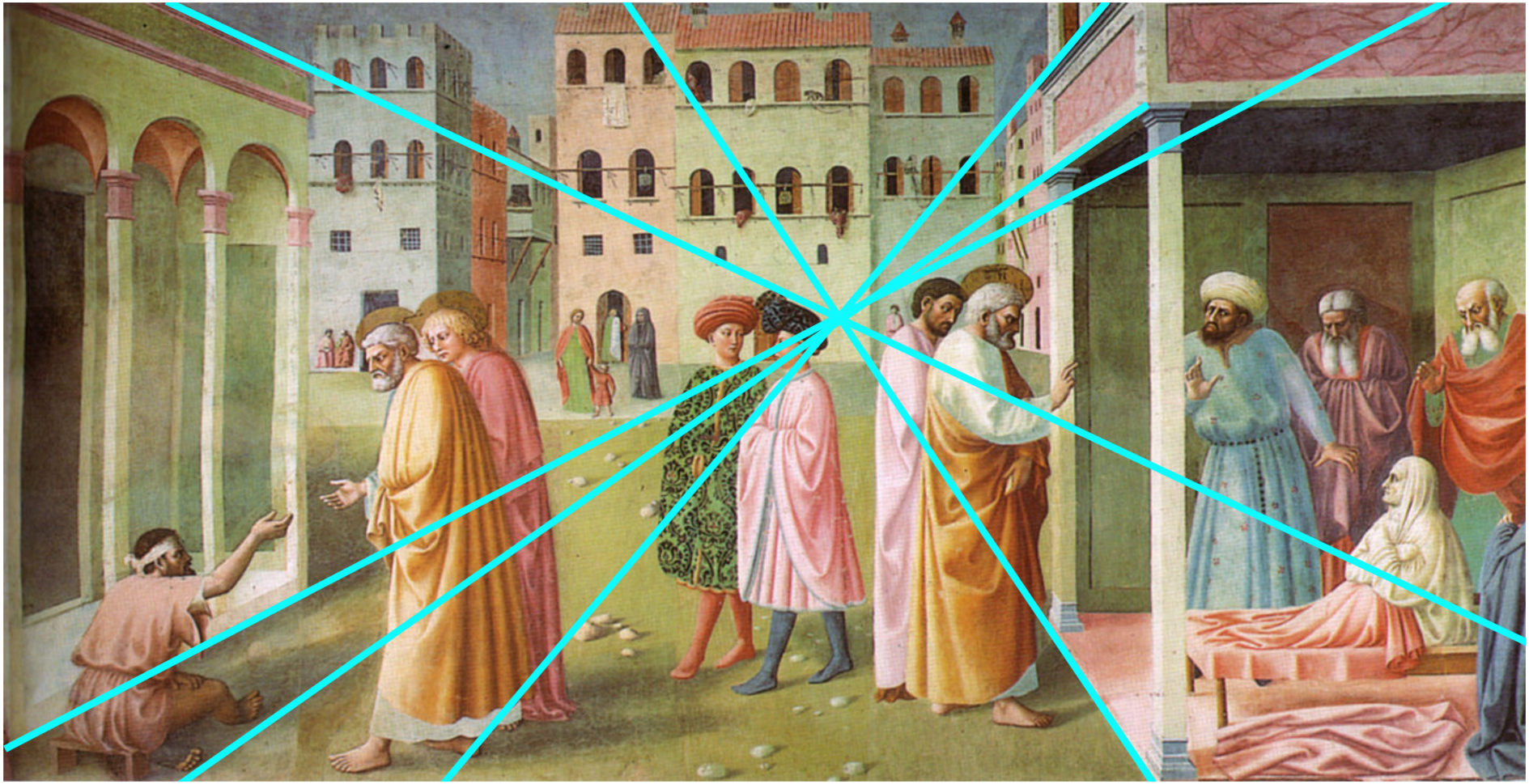

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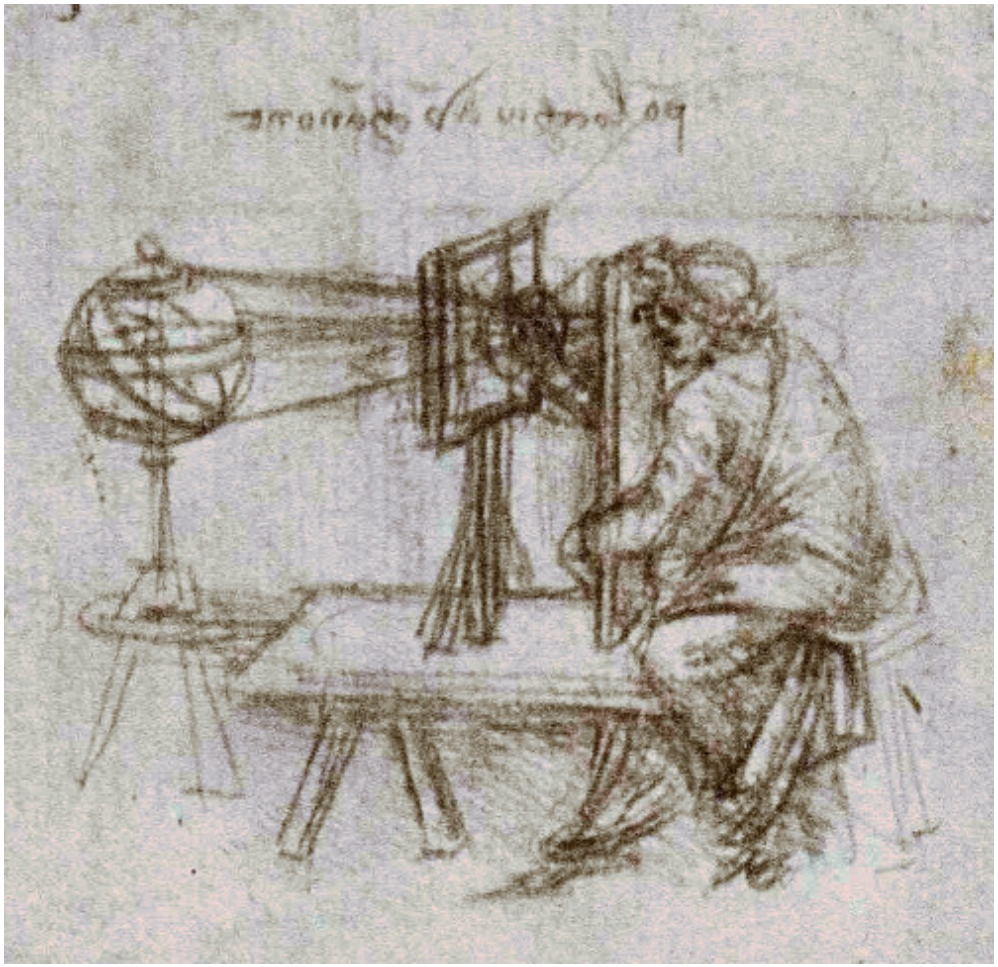
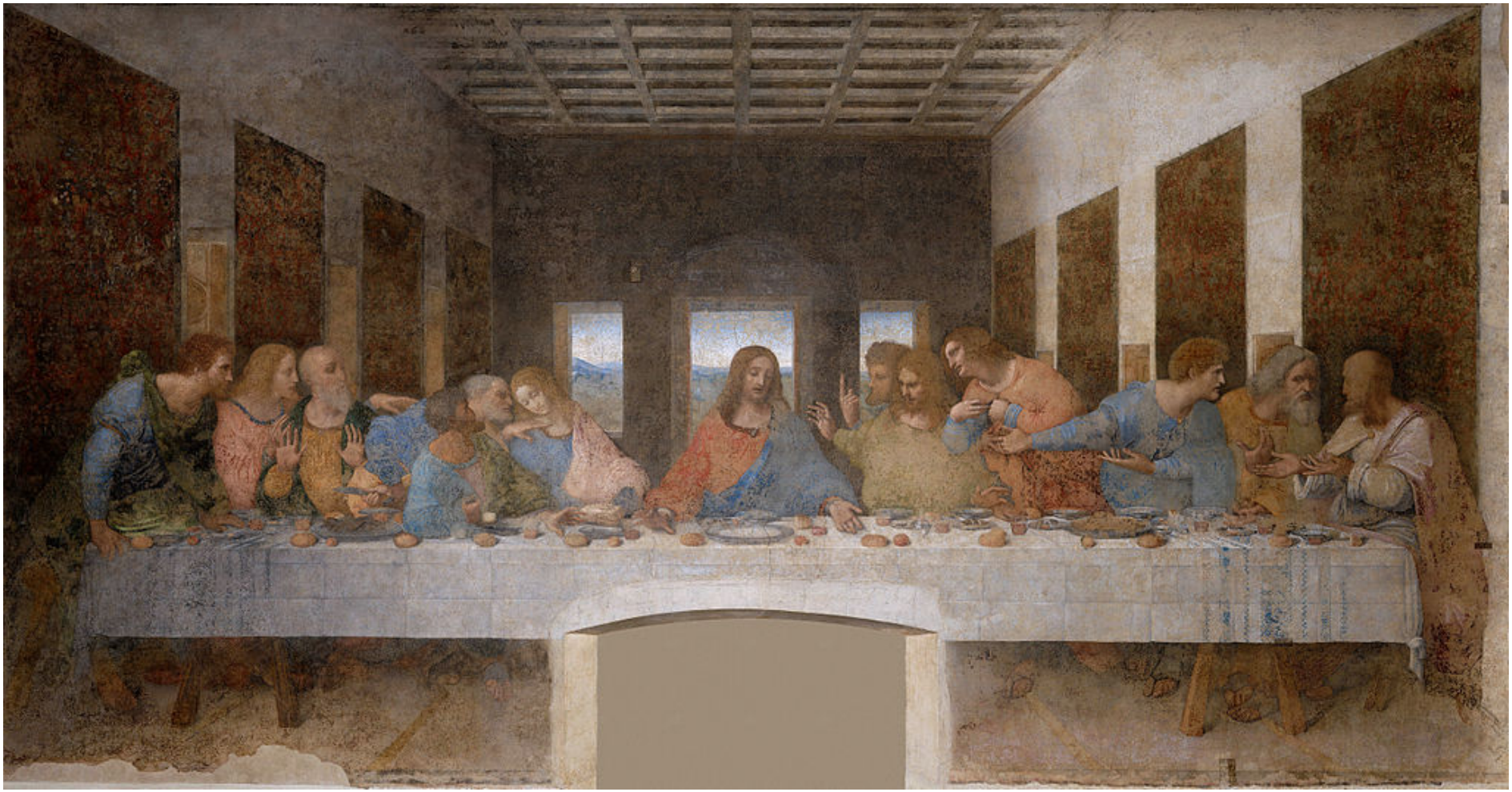
Da Vinci's Perspectograph



Perspectograph by Leonardo Da Vinci (1452–1519). In *Codex Atlanticus*.

Da Vinci Example



Last Supper (1495–1498) by Leonardo Da Vinci.

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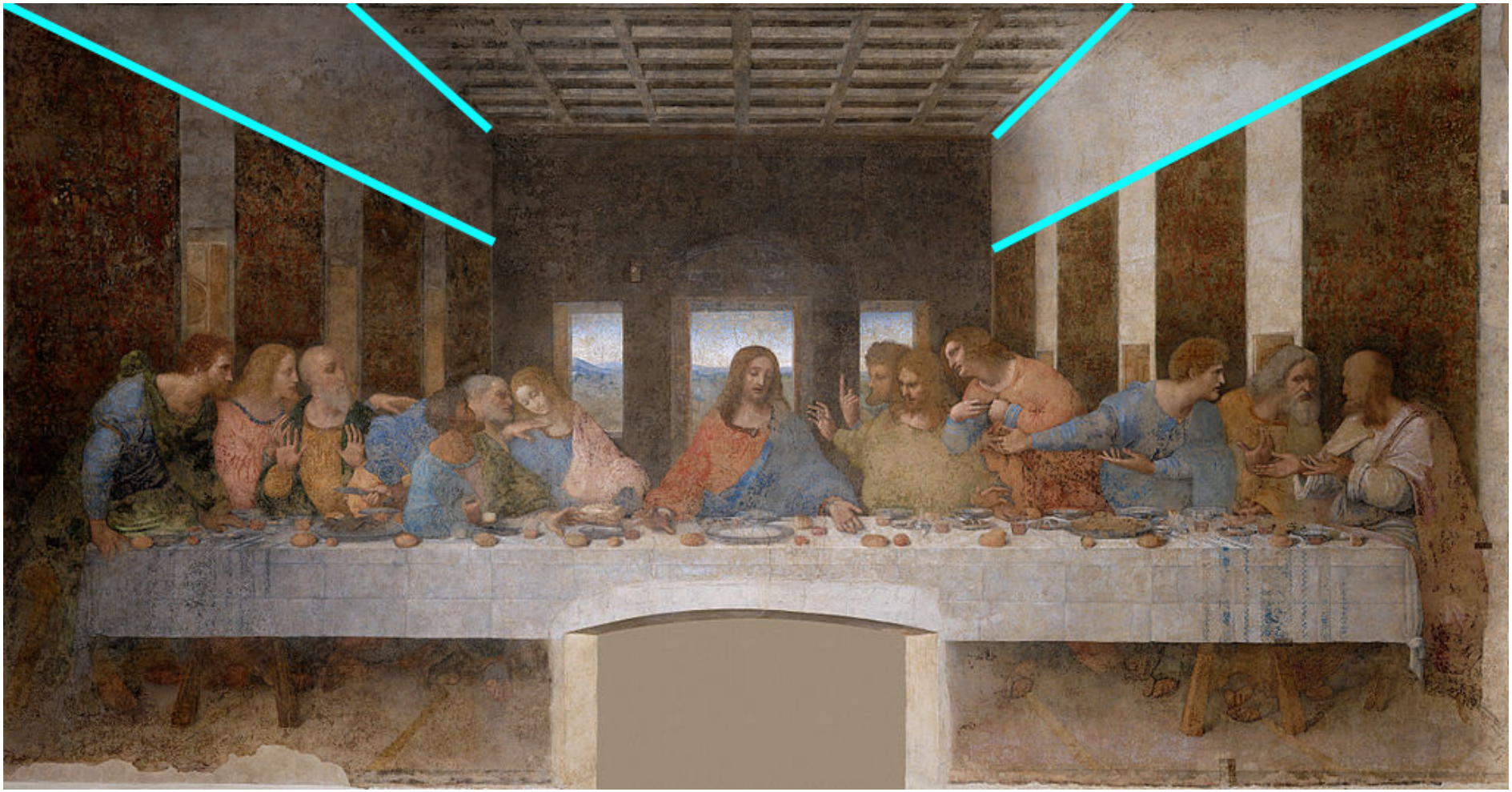

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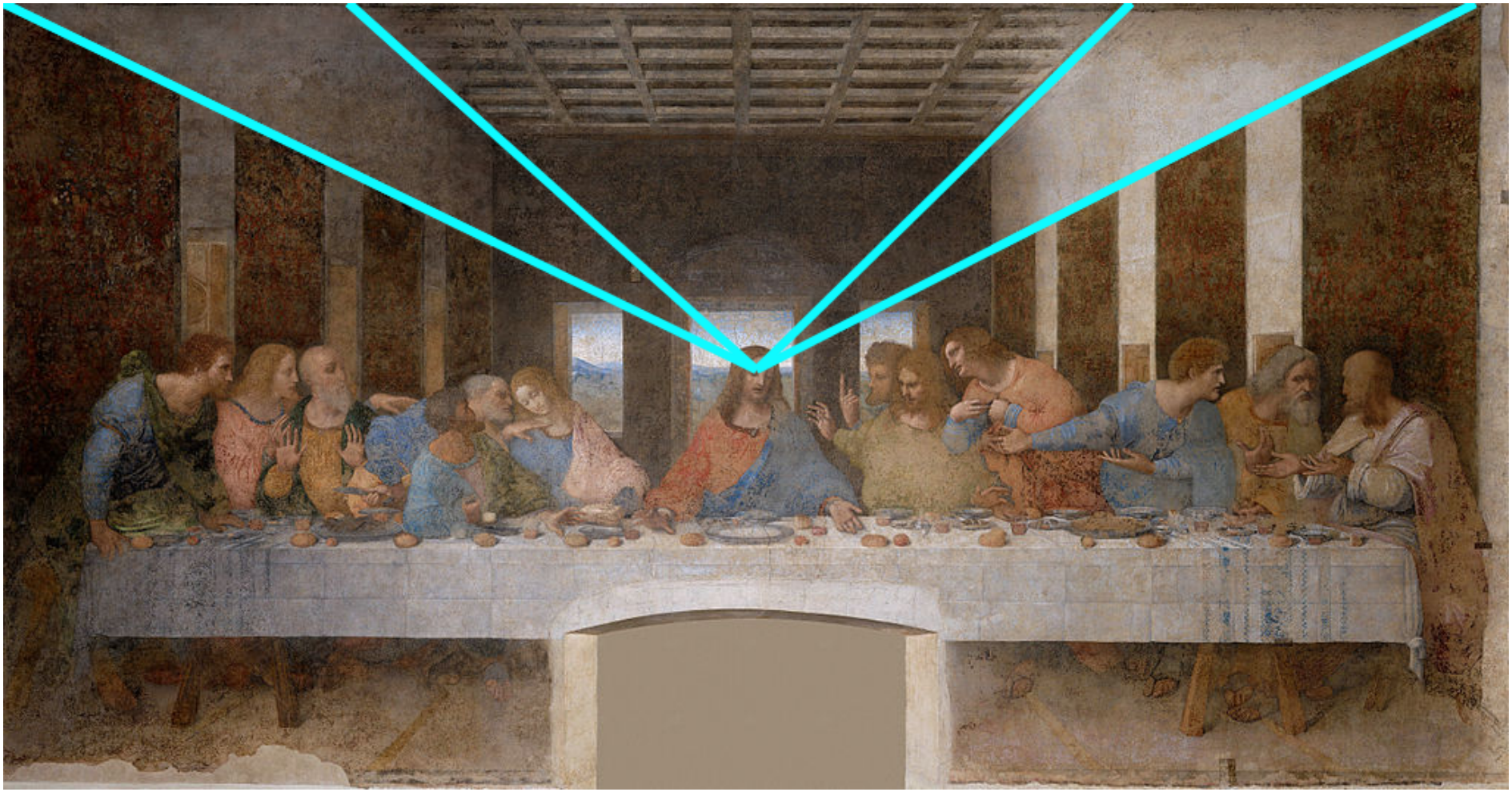

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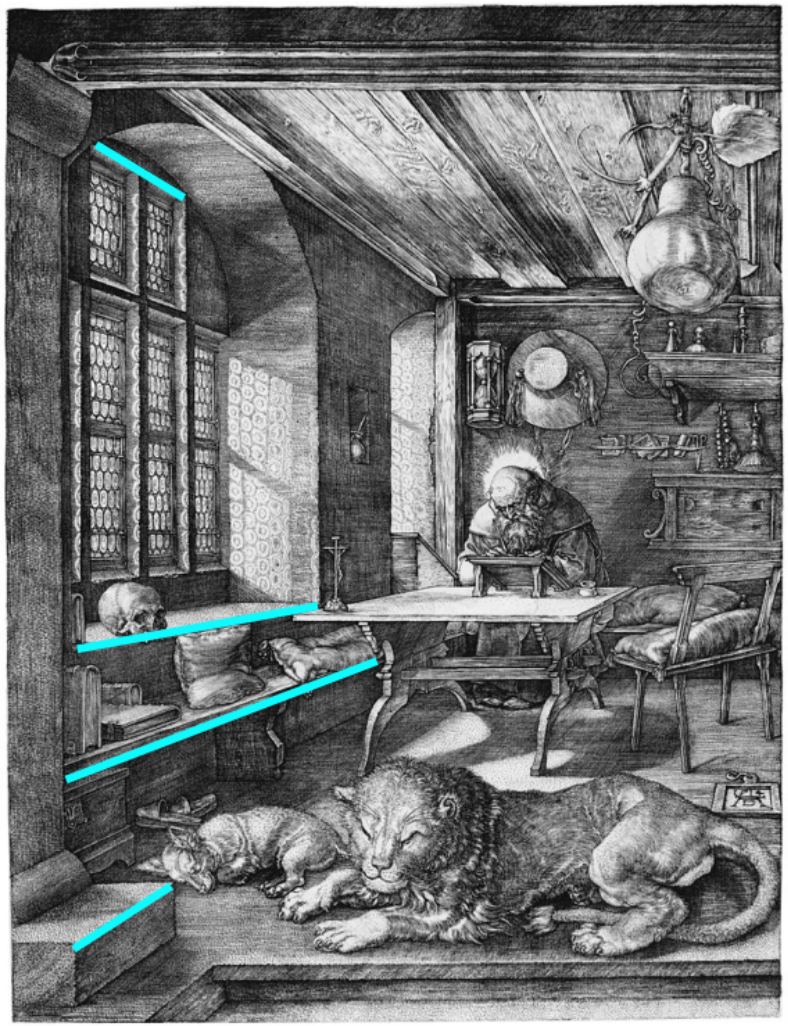
European Spread

From *Underweysung der Messung* (1525) by Albrecht Dürer.

Dürer Example

Saint Jerome in his Study (1514) by Albrecht Dürer.

Dürer Example



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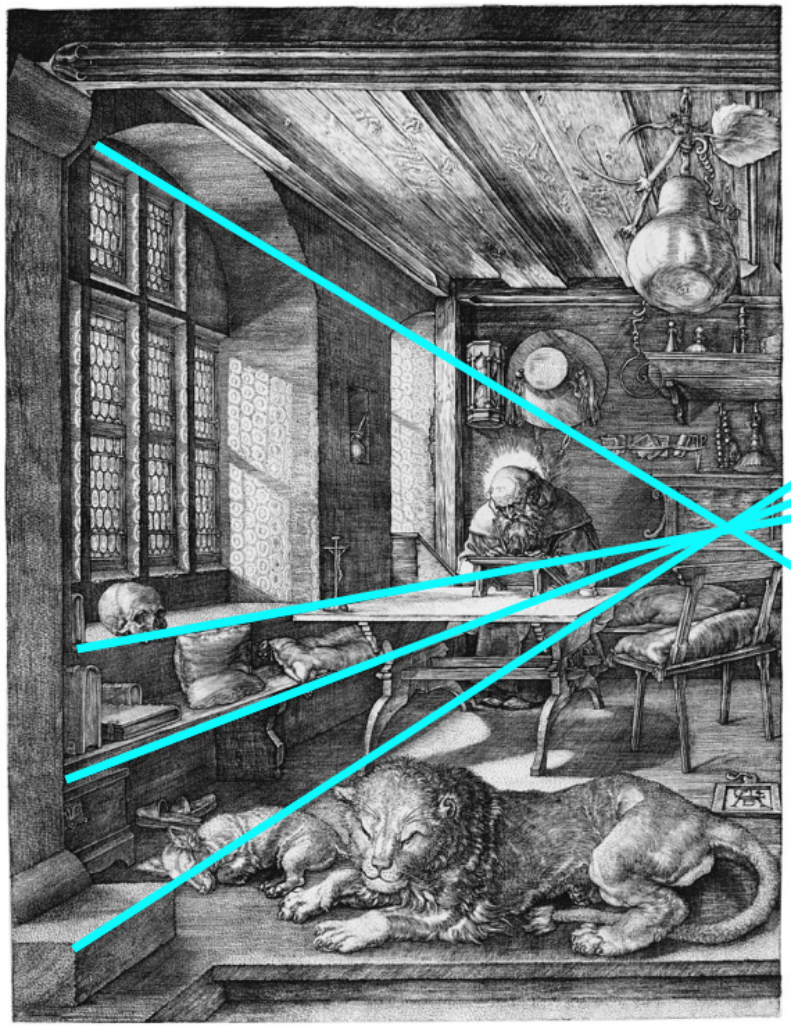

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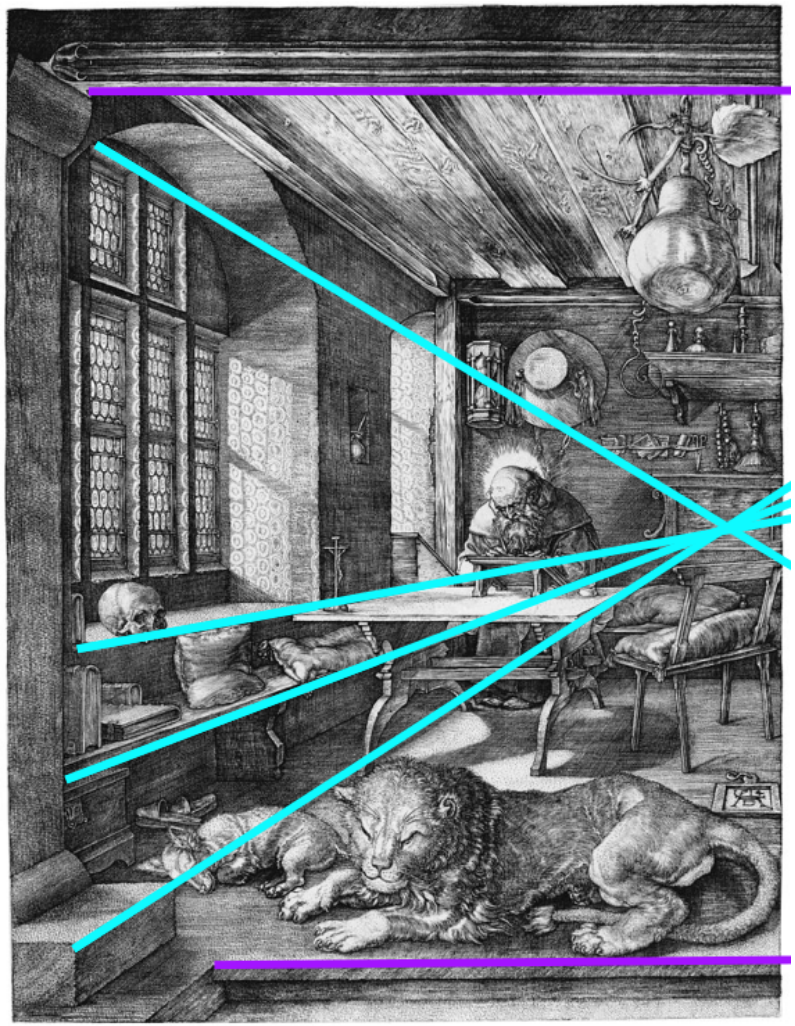

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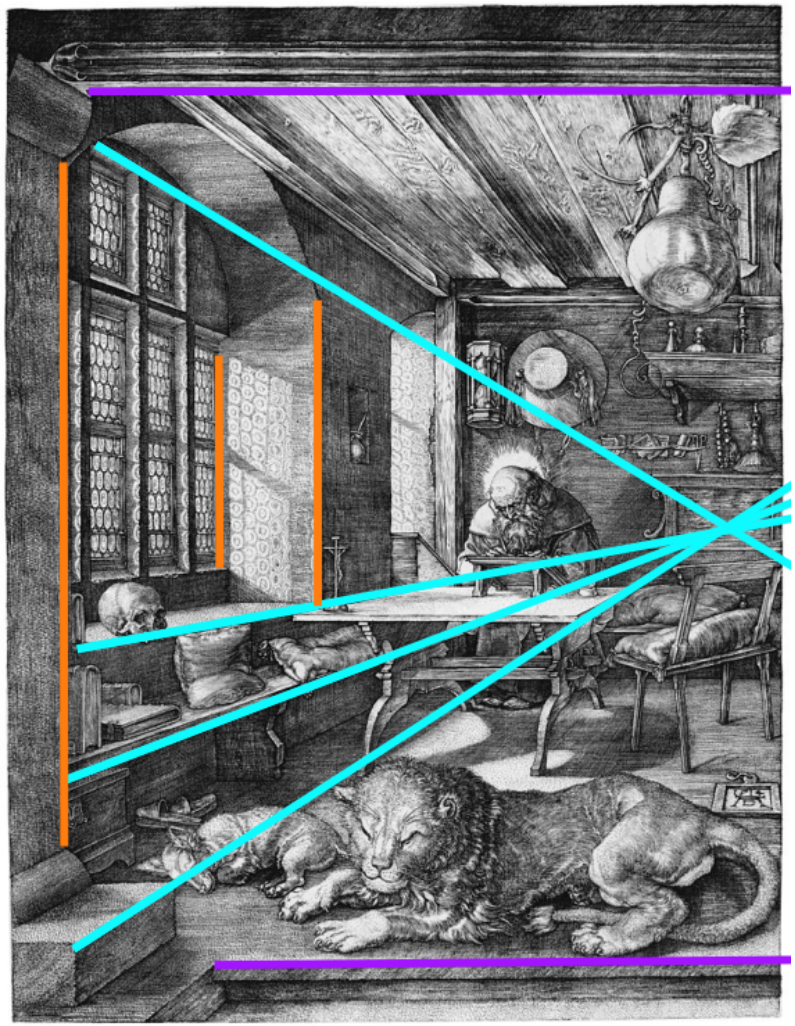

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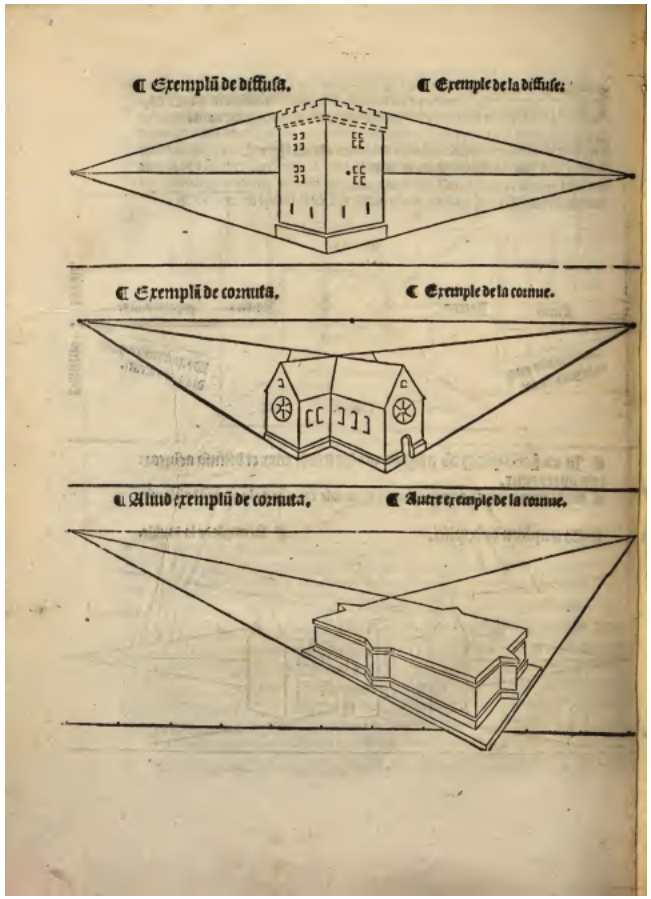
Multi-point Perspective



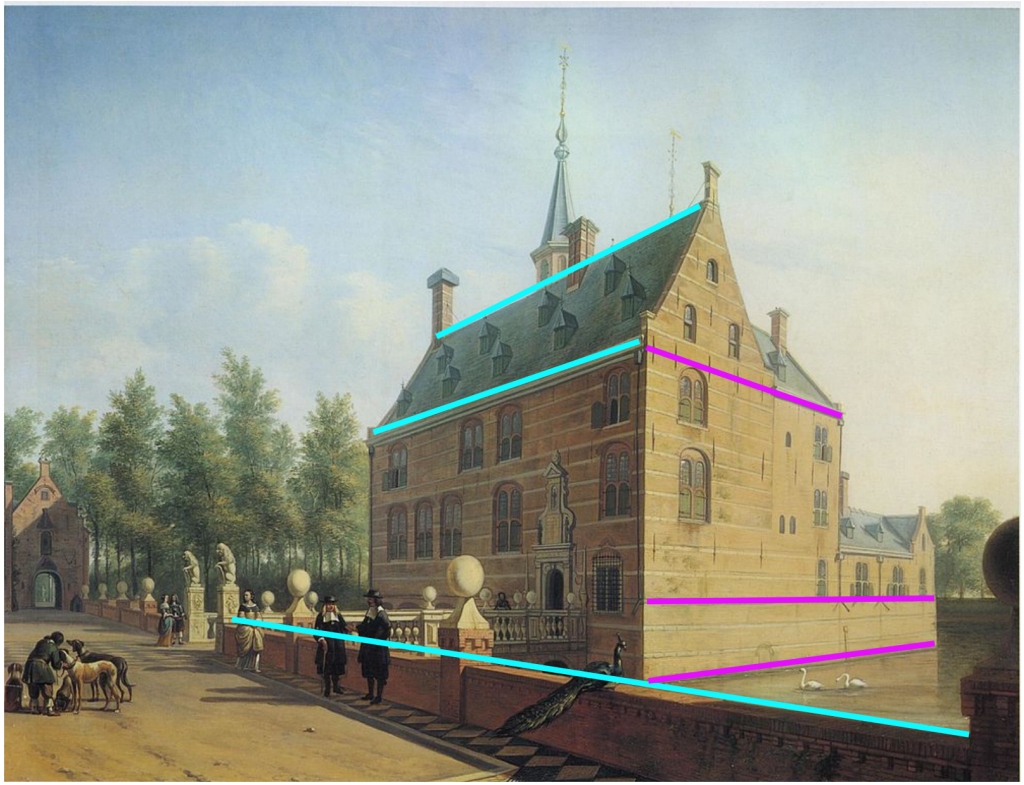
From *De Artificiali Perspectiva* (1505) by Jean Pèlerin "Viator."

As linear perspective spread through Europe, mathematicians and artists figured out how to treat multiple vanishing points.

Dutch Golden Age

View of the Castle in Heemstede, North Holland (1667) by Gerrit Berckheyde.

Dutch Golden Age



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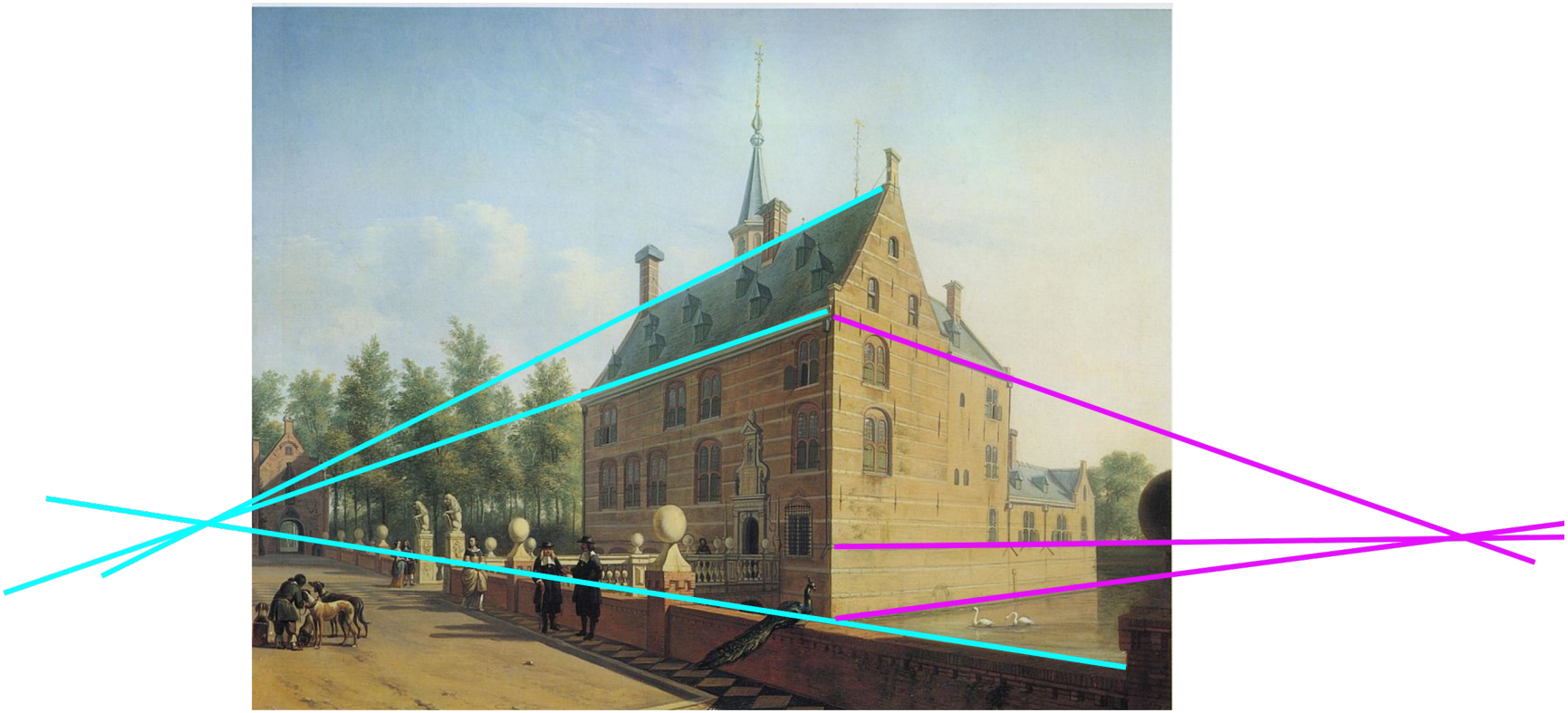

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Eastward Spread

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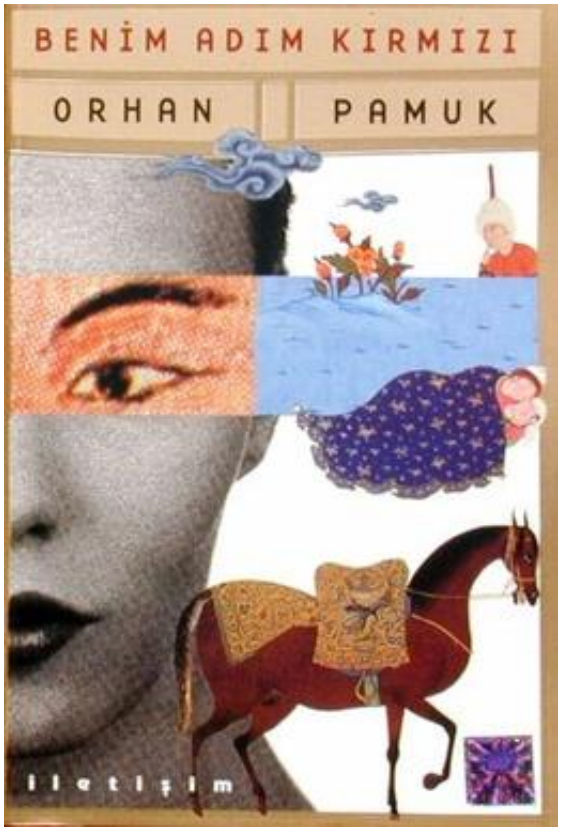
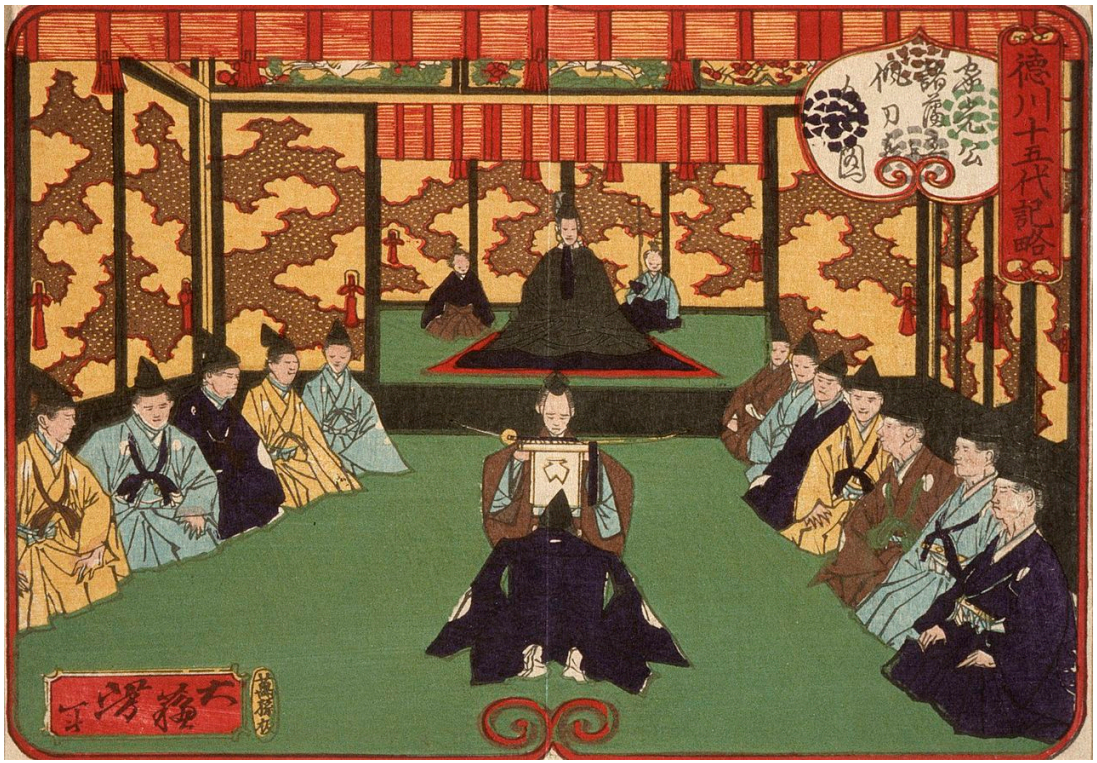


Nobel Prize winning writer Orhan Pamuk gives a fictionalized account in *My Name is Red* (2001), a murder mystery set in the Ottoman Empire in the late 1500s, involving artists secretly working on Renaissance-style art for a book comissioned by the Sultan.

Eastward Spread

The Tokugawa Shogunate in Japan enforced extreme isolationism from the 1630s until 1853.



Tokugawa Iemitsu Receiving Lords in Audience (1875) by Tsukioka Yoshitoshi (月岡 芳年).

Eastward Spread

The Japanese did, however, maintain limited interactions with the Dutch.

Painting of Dejima in Nagasaki Bay (c. 1820).

Eastward Spread

Japanese scholars studied Dutch books that trickled into Japan.

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Pictures in linear perspective, called *uki-e* (浮絵), began appearing in the late 1730s.

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Outline

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Let's look at the geometry to explain this.

Parallel and Perspective Projections

Parallel and perspective projections are closely related!

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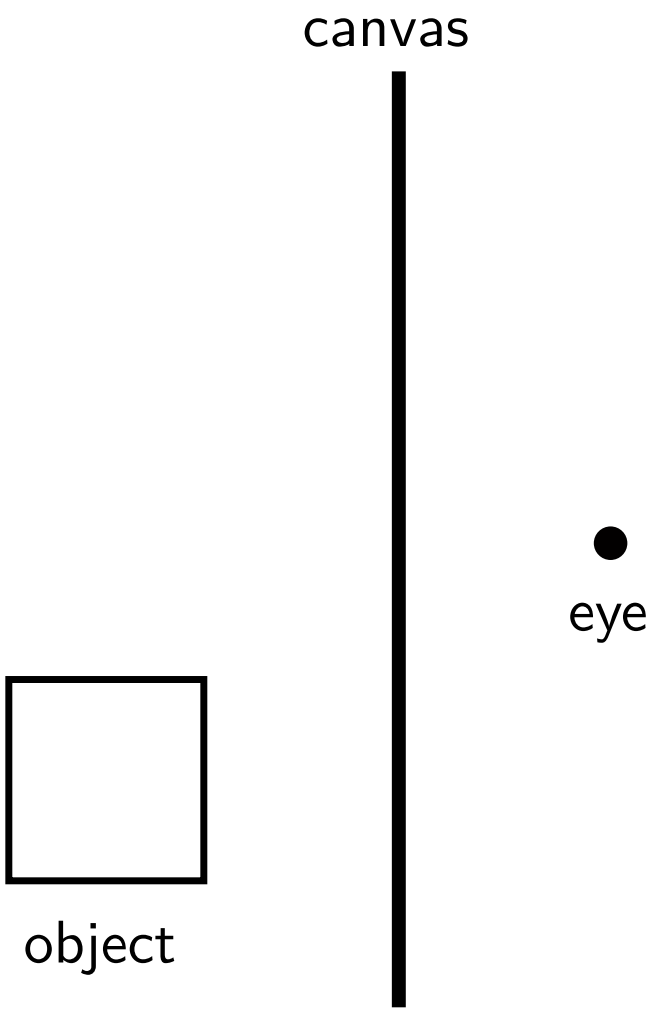
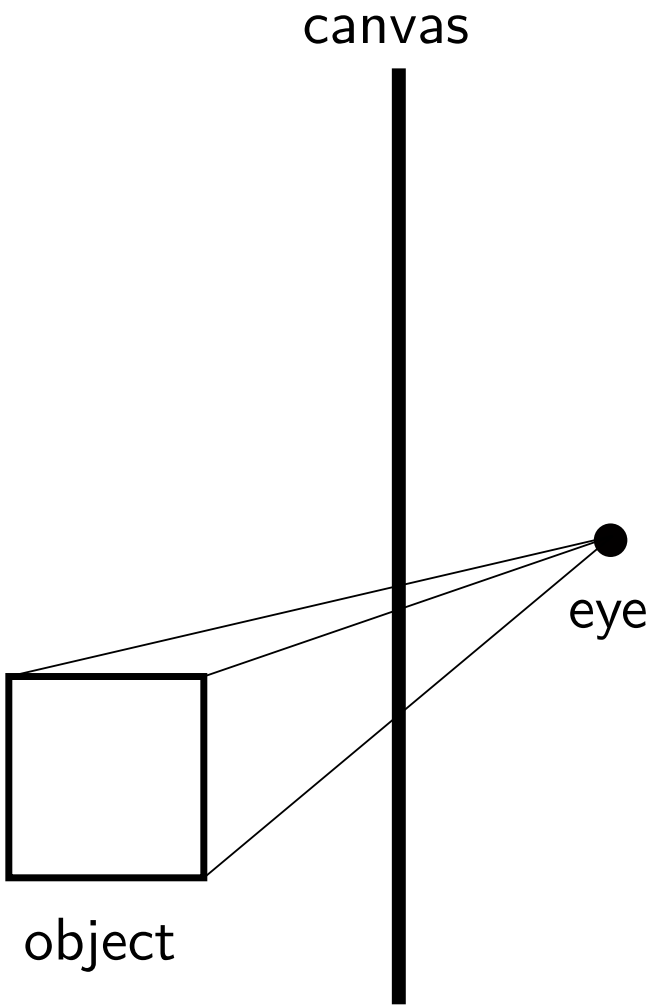
Parallel projection is a limit of perspective projections as the center of projection (ie, the eye) tends off to infinity along a line. This line specifies the projection direction of the parallel projection.

Limit of Perspective Projections



Limit of Perspective Projections



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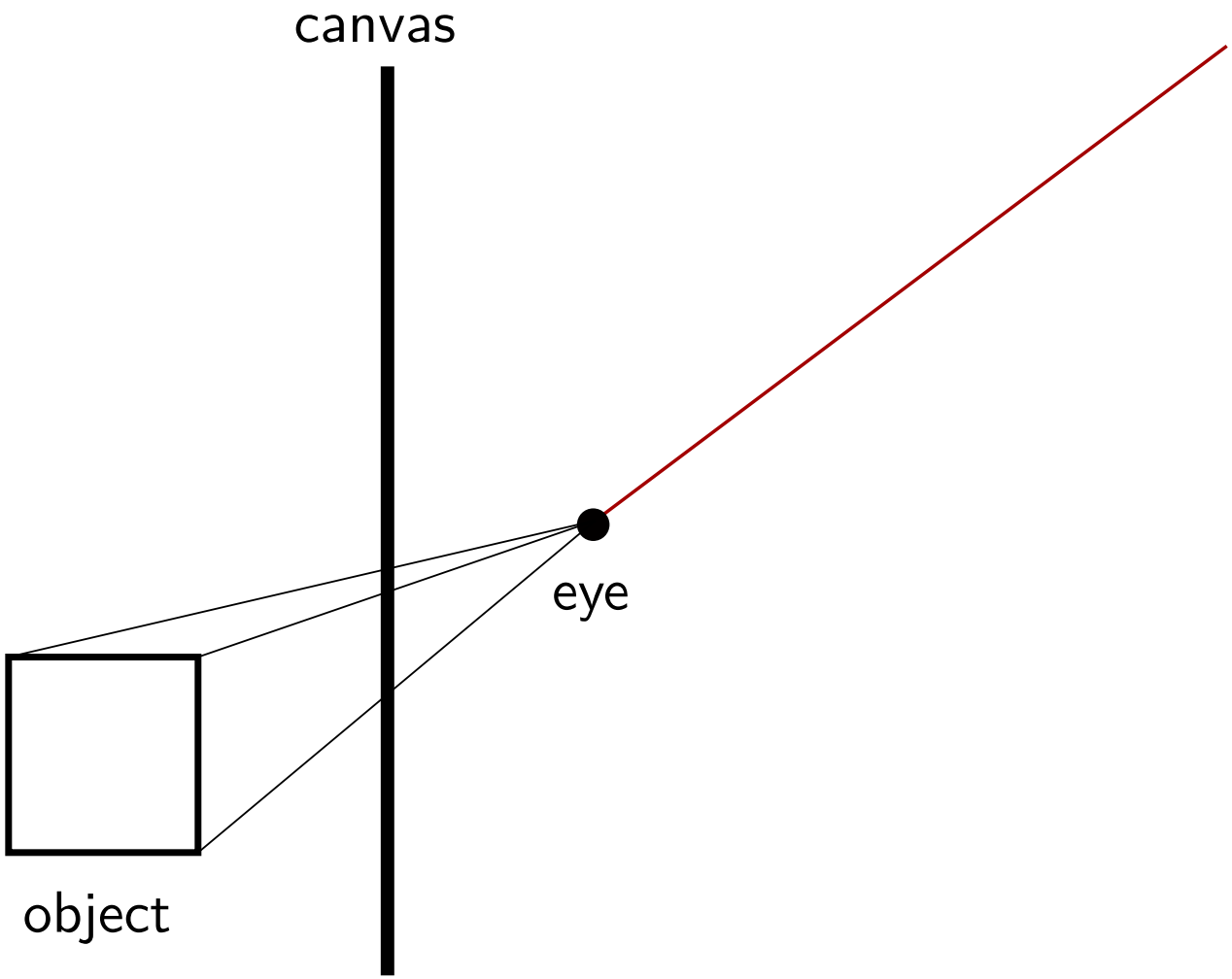

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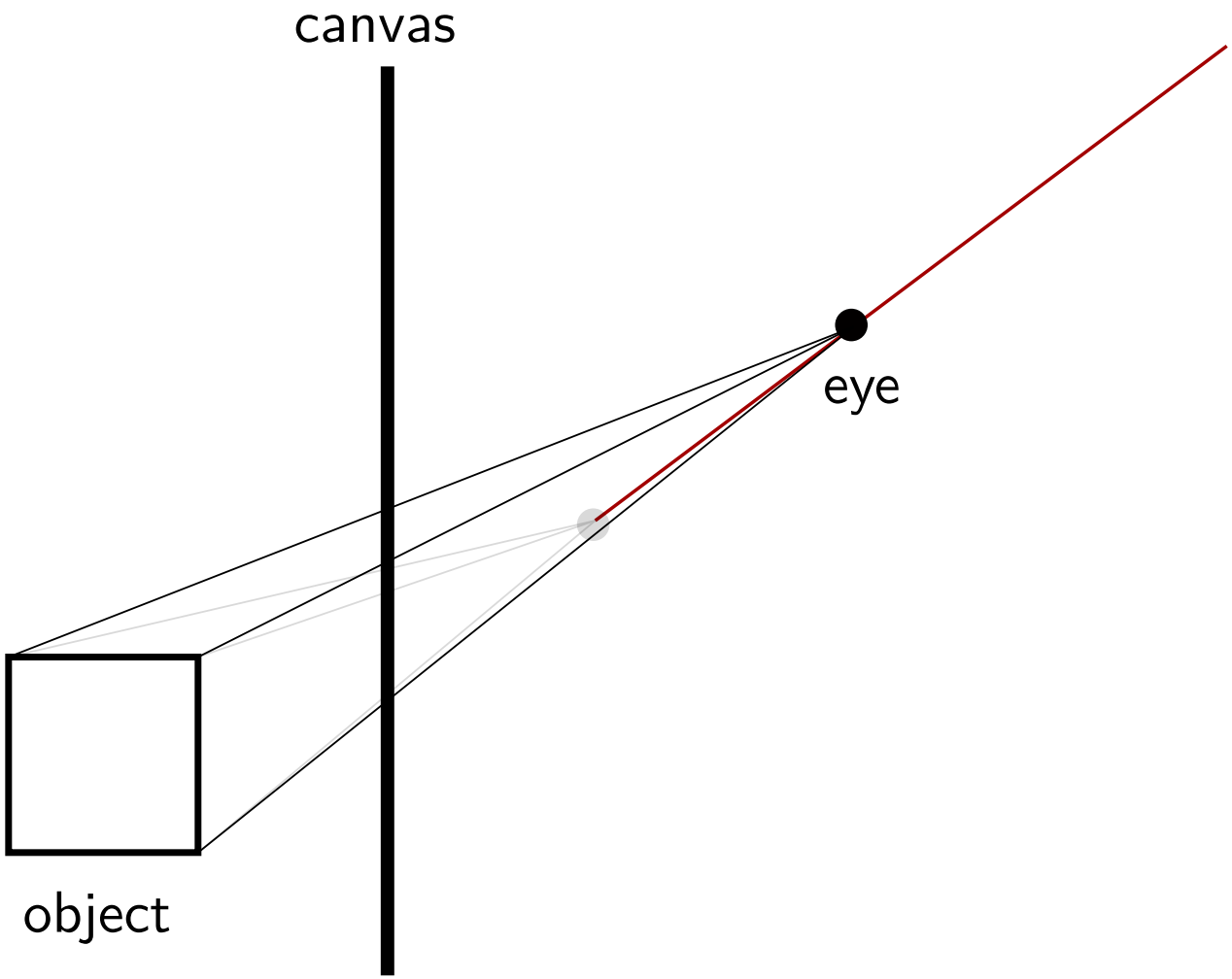

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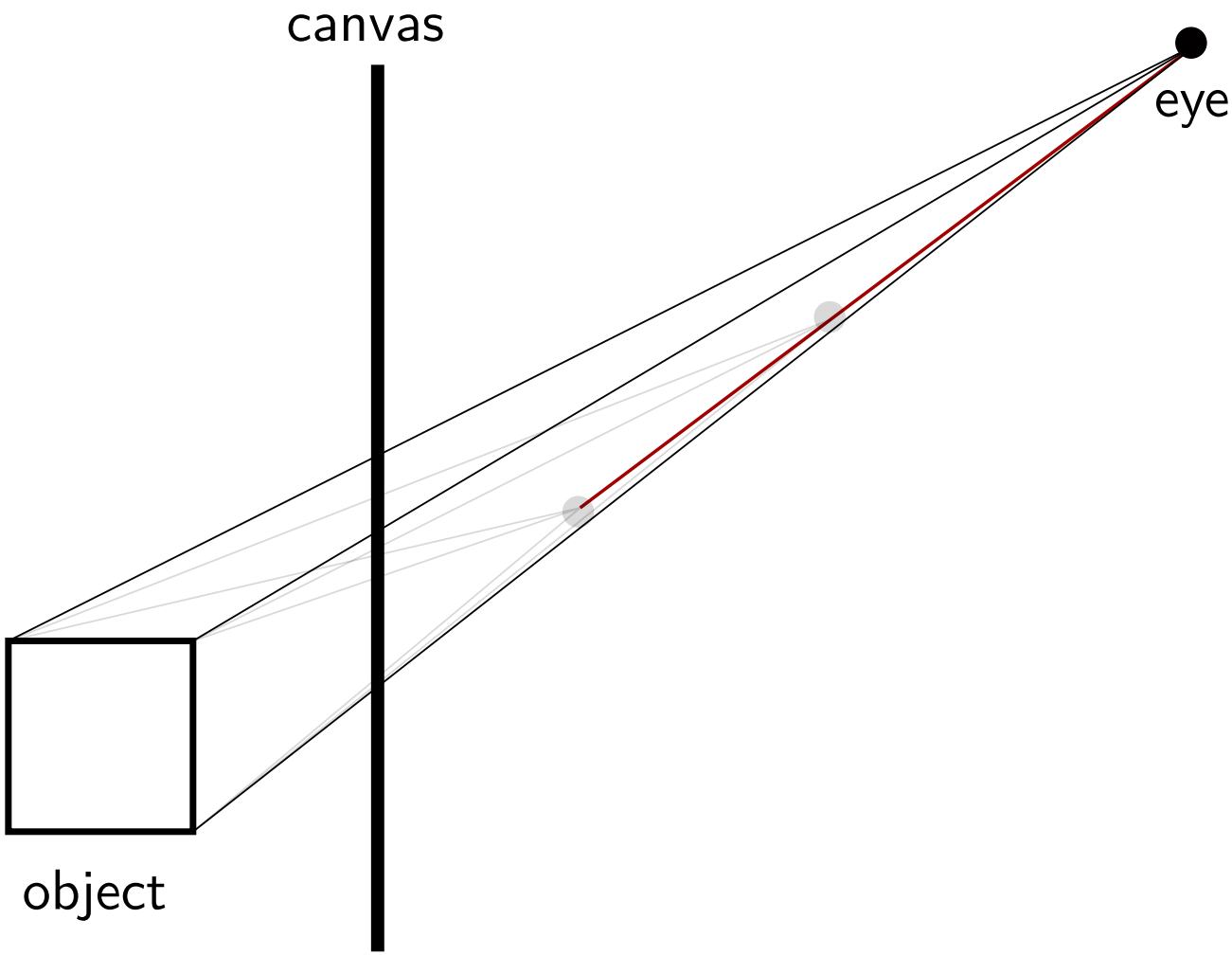

Infinitely Long Arms

In other words, parallel perspective is what would happen if you had infinitely long arms and painted in linear perspective.

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Unlike linear perspective, a painting in parallel perspective isn't tied to any particular vantage point.

This makes parallel perspective a natural choice for large paintings that you physically can't view all at once (such as East Asian handscrolls!).

Qing Dynasty Handscroll

Along the River During the Qingming Festival Season (1736) by Chen Mei (陳枚), Sun Hu (孫祜), Jin Kun (金昆), Dai Hong (戴洪), and Cheng Zhidao (程志道).

Wikimedia Commons Link

“Dolly Zoom”

The focal distance of our eye (ie, distance from pupil to retina) never changes, so we can’t see in parallel perspective, and it’s hard to imagine what changing the focal distance looks like.

"Dolly Zoom"

The focal distance of our eye (ie, distance from pupil to retina) never changes, so we can't see in parallel perspective, and it's hard to imagine what changing the focal distance looks like.

But you *can* change the focal distance on a camera!

Computer generated Dolly Zoom: Wikimedia Commons Link

Alfred Hitchcock's *Vertigo* (1958): YouTube Link

Projective Space

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Projective geometry provides a framework for treating parallel and perspective projections in a neat, unified way.

These ideas continue to play an important role in modern mathematics, including in algebraic geometry (that's what I do!).

Thank you!

Further Reading

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