From category $\mathcal{O}^{\infty}$ to locally analytic representations

Joint work with Matthias Strauch
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$F/\mathbb{Q}_p$ a finite extension

1 Locally analytic representations

Let G be a locally analytic group / F

Ex. If $\mathbb{G}$ is an algebraic group / F, then $G := \mathbb{G}(F)$ is a locally analytic group.
Closed subgroups of this (in the p-adic topology) are also locally analytic groups.

barrelled locally convex vector space

A locally analytic representation of G is a 'reasonable' topological vector space V over F with a continuous action of G such that, for any $v \in V$, the orbit map
$$o_v : G \longrightarrow V$$
is a locally analytic function on G.

$G \times V \to V$ is continuous

better: finite extn E/F for coefficients

Examples:
- Any smooth representation (o_v is locally constant)
- G reductive and V is finite dim'l algebraic rep
- $G = GL_2(F)$ and V is 1-forms on Drinfeld's "p-adic upper half plane" ($= \mathbb{P}^1 \setminus \mathbb{P}^1(F)$)
- (*) $G = GL_2(\mathbb{Q}_p)$ and V is the subspace of locally analytic vectors in the Banach space representation attached to a 2-dimensional p-adic Galois representation via Colmez's p-adic Langlands correspondence
- Etc.

Montréal functor

$\left\{ \begin{array}{l} \text{admissible unitary Banach reps of } GL_2(\mathbb{Q}_p) \\ \text{over } F \text{ which have a central character} \\ \text{and which are residually of finite length} \end{array} \right\}$

$\downarrow$

$\left\{ \begin{array}{l} \text{2-dim'l reps of } Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \text{over } F \end{array} \right\}$

which, roughly, induces bijection between iso classes of absolutely irreducible objects

Goal: Produce a class of locally analytic representations that can be understood using Lie algebra representations

[Turns out that the class we'll produce is related to (*) for semistable non-crystalline Galois reps]

Algebraic perspective on locally analytic reps:

Breuil constructs a Banach rep $B(k,\mathcal{L})$ corresponding

$D(G) = C(G)'$ is distributions on G

continuous dual of locally analytic functions on G

This is naturally a topological F-algebra

not commutative, not noetherian, is Fréchet-Stein if G is compact

- $F[G]$ is a dense subalgebra of $D(G)$ — group algebra

$$g \longmapsto \left[\delta_g : f \longmapsto f(g)\right] \in D(G)$$

- If $\mathfrak{g} = \mathrm{Lie}(G)$, then $U(\mathfrak{g})$ is also a subalgebra of $D(G)$

$$x \cdot f = \lim_{t \to 0} \frac{\exp(tx) f - f}{t}$$

$$x \longmapsto \left[f \longmapsto ((-x)\cdot f)(1)\right] \in D(G)$$

If V is a locally analytic representation, then

$$\delta_g \cdot v = g \cdot v$$

extends to a separately continuous $D(G)$-module structure on V.

Thm (Schneider-Teitelbaum)

$$\begin{tikzcd}
\left\{\begin{matrix}\text{locally analytic} \\ \text{representations on} \\ \text{vector spaces of} \\ \text{compact type}\end{matrix}\right\}^{op} \arrow[r, "\sim"] & \left\{\begin{matrix}\text{separately continuous} \\ D(G)\text{-modules on} \\ \text{nuclear Fréchet} \\ \text{vector spaces}\end{matrix}\right\} \\
\left\{\begin{matrix}\text{admissible locally} \\ \text{analytic reps}\end{matrix}\right\}^{op} \arrow[u] \arrow[r, "\sim"] & \left\{\begin{matrix}\text{coadmissible} \\ D(G)\text{-modules}\end{matrix}\right\} \arrow[u]
\end{tikzcd}$$

$$V \longmapsto V'$$

a kind of finiteness condition (eg, finitely presented $\Rightarrow$ coadmissible when G is compact)

If $H \subseteq G$ is a closed subgroup, can consider subring $D(\mathfrak{g}, H)$ (not closed) generated by $U(\mathfrak{g})$ and $D(H)$.

Locally analytic reps of H that are "compatibly" $\mathfrak{g}$-modules are naturally $D(\mathfrak{g}, H)$-modules.

Two actions of $\mathfrak{h} = \mathrm{Lie}(H)$ agree, & $\delta_h(xm) = \mathrm{Ad}(h)(x)(h.m)$ for $h \in H$, $x \in \mathfrak{g}$, $m \in M$.

More about all of this:

- Schneider-Teitelbaum '01, '02, '03, '05
- Kohlhaase '07
- Schmidt-Strauch '16
- Emerton '17
- [AS] sections 6 & 7
- ⋮

② Outline of the construction

Notation

- G a split reductive group / F
- T a split maximal torus

to semistable non-cryst. Galois reps. It's a completion of a locally an rep $\Sigma(k, \mathcal{L})$, which in turn is a subquotient of

$$\mathrm{Ind}_P^G(\sigma(k, \mathcal{L})),$$

which is in the image of our functor.

When $k = 2$...

Let N be $\mathfrak{p}$-module

$$\begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} 0 & x-y \\ 0 & 0 \end{pmatrix}$$

and $M = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(N)$. Then

$$\mathrm{Ind}_P^G(\sigma(2, \mathcal{L})) = \mathcal{F}_P^G(M, \mathbb{1})$$

where $\mathcal{L}$ on left corresp. to choice of log on right.

· P a parabolic containing T
· $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{p} = \mathrm{Lie}(P)$

BGG category $\mathcal{O}^{\mathfrak{p}}$
(1) Finitely generated as $\mathfrak{g}$-module
(2) locally finite dimensional as $\mathfrak{p}$-module
(3) weights defined over F
(4) Action of $\mathfrak{t}$ is semisimple ⟵ not stable under extensions!

Extension closure of $\mathcal{O}^{\mathfrak{p}}$ is "thick" BGG category $\mathcal{O}^{\mathfrak{p},\infty}$ defined by conditions (1), (2), & (3) [No (4)]

Write $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$, $\mathcal{O}^{\mathfrak{p},\infty}_{\mathrm{alg}}$ for subcategories of $\mathfrak{g}$-modules which have algebraic weights

Thm. There is a bifunctor

[AS, 4.2.3—4]

$$\mathcal{O}^{\mathfrak{p},\infty}_{\mathrm{alg}} \times \left\{\begin{matrix}\text{smooth strongly} \\ \text{admissible reps} \\ \text{of } P\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{admissible locally} \\ \text{analytic reps} \\ \text{of } G\end{matrix}\right\}$$
$$(M,V) \longmapsto \mathcal{F}^G_P(M,V)$$

which is contravariant in M, covariant in V, and exact in both.

Orlik-Strauch '14: $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ in the first entry, any admissible smooth rep of P in the second.

Why extend from $\mathcal{O}^{\mathfrak{p}}$ to $\mathcal{O}^{\mathfrak{p},\infty}$?

- $\mathcal{O}^{\mathfrak{p}}$ not being stable under extensions can be technically inconvenient

- $\mathfrak{g}$-modules that are not in category $\mathcal{O}^{\mathfrak{p}}$ do come up!

Ex. $G = SL_2$ $\quad P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ $\quad X = G/P = \mathbb{P}^1$

There are five indecomposable D-mods on X

$$D/\partial \qquad D/t \qquad D/\partial t \qquad D/t\partial \qquad D/t\partial t$$

Taking $\Gamma(X,-)$ is an equivalence (Beilinson-Bernstein localization)

$$\text{D-Mod} \longrightarrow \mathfrak{g}\text{-Mod}_0$$

and

$$\Gamma(X, D/t\partial t) \in \mathcal{O}^{\infty} \setminus \mathcal{O}$$

Ex. Representations of form $\mathcal{F}^G_P(M,V)$ w/ $M \in \mathcal{O}^{\infty} \setminus \mathcal{O}$ are related to reps that occur in work of Breuil & Schraen in the p-adic Langlands program. [AS, section 5]

To construct $\mathcal{F}^G_P(M,V)$...

choice of logarithm doesn't matter for $M \in \mathcal{O}^{\mathfrak{p}}$

(a) Use a p-adic logarithm to construct a locally analytic action of P on M lifting action of $\mathfrak{p}$. M with this action of P is $\mathrm{Lift}(M, \log)$. This is a $D(\mathfrak{g}, P)$-module

cf. ③

(b) V' is a $D(P)$-module on which $\mathfrak{p}$ acts trivially, so can regard it is a $D(\mathfrak{g}, P)$-module with a trivial action of $\mathfrak{g}$.

(c) $\mathrm{Lift}(M, \log) \otimes V'$ is a $D(\mathfrak{g}, P)$-module

even though $D(\mathfrak{g}, P)$ is not a Hopf algebra cf ④

(d) Define

$$\check{\mathcal{F}}^G_P(M, V) = D(G) \otimes_{D(\mathfrak{g}, P)} \left[\mathrm{Lift}(M, \log) \otimes V'\right]$$

and

$$\mathcal{F}^G_P(M, V) = \left[\check{\mathcal{F}}^G_P(M, V)\right]'$$

③ Lifting Lie algebra representations

[AS, section 2]

Let P be a connected algebraic group $/F$ with split maximal torus T and unipotent radical U.
Let $\mathfrak{p} = \mathrm{Lie}(P)$, $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{u} = \mathrm{Lie}(U)$.

Let M be a $\mathfrak{p}$-module such that
- Locally finite dimensional as $\mathfrak{p}$-module
- Locally nilpotent action of $\mathfrak{u}$
- Algebraic weights

Eg, the $\mathfrak{p}$-module underlying any object of $\mathcal{O}^{\mathfrak{p},\infty}_{\mathrm{alg}}$

We'll construct an action of P on M which "lifts" the original action of $\mathfrak{p}$

differentiates to

Until further notice, M is finite dimensional.

We'll gradually increase the "complexity" of P.

(i) If $P = U$, we have group isomorphisms

$$\begin{tikzcd} U \arrow[r, bend left, "\log"] & \mathfrak{u} \arrow[l, bend left, "\exp"] \end{tikzcd}$$

set of nilpotent linear maps $M \to M$

Since the action of $\mathfrak{u}$ on M is nilpotent, we can exponentiate it, and

$$U \xrightarrow{\log} \mathfrak{u} \longrightarrow \mathrm{Nil}(M) \xrightarrow{\exp} GL(M)$$

is an action of U lifting the original action of $\mathfrak{u}$.

important case!

(ii) Suppose $P = T$.

Since we're over a p-adic field, have $\exp: \mathfrak{t} \dashrightarrow T$ but there exist homomorphisms $T \to \mathfrak{t}$ which invert exp. These are logarithms on T.

Ex. $T = \mathbb{G}_m$ $F = \mathbb{Q}_p$
Maps $\mathbb{Q}_p^\times \to \mathbb{Q}_p$ inverting exp are determined by where p goes.
Choosing $p \mapsto 0$ gives the "Iwasawa log."

If $\mathfrak{t}$ acts nilpotently on M, can now use a log on T to construct action of T on M just as before:

$$T \xrightarrow{\log} \mathfrak{t} \longrightarrow \mathrm{Nil}(M) \xrightarrow{\exp} GL(M)$$

Ex. $T = \mathbb{G}_m$

2 dim'l

$$M \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

action of $1 \in \mathfrak{t}$

Then can lift the action of $\mathfrak{t} = F$ to an action of $T = F^\times$ using the map $\rho: F^\times \to GL_2(F)$

$$\rho(a) = \begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}$$

[AS, 2.3.2-3] That deals with the case where 0 is the only weight. For other weights, we "twist", and if there's more than one weight, we just lift one generalized weight space at a time.

all weights are still assumed to be algebraic!

Ex. $T = \mathbb{G}_m$

$$M \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Can lift using $\rho: F^\times \to GL_2$ given by

$$\rho(a) = \underbrace{a^2}_{\text{character of } T \text{ that differentiates to the weight 2}} \underbrace{\begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}}_{\text{what we had above}}$$

Let $\mathrm{Lift}(M, \log)$ be this locally analytic rep'n of T.

eg, the $\mathfrak{t}$-module underlying an obj in category $\mathcal{O}^{\mathfrak{t}}$!

Note that the log comes up in dealing with non-semisimplicity. If the Lie algebra rep is semisimple (ie, generalized weight spaces are just weight spaces), the log doesn't matter.

(iii) P semisimple

Jantzen does this in "Representations of alg. groups" §7. Basically, idea is just to glue actions of root subgroups (which are unipotent) with action of torus.

[AS, 2.4.8]

Since P semisimple, action of $\mathfrak{t}$ is forced to be semisimple, so the lifted action is even algebraic (no logs show up!)

(iv) P reductive

[AS, 2.4.9]

Let $P' = [P,P]$ be derived subgroup. Get an action of P' using (iii). Have action of T from (ii) if we choose a logarithm. Actions are compatible, and P is generated by T & P', so done.

agree on $T \cap P'$, and compatible w/ conjugation $T \circlearrowright P'$

(v). P general

[AS, 2.4.11]

Let L be a Levi subgroup so that $P = U \rtimes L$. L is reductive, so get action using (iv). U unipotent, so get an action using (i). Again compatible, so get an action of P.

Upshot: For any finite dimensional $\mathfrak{p}$-module M on which $\mathfrak{u}$ acts nilpotently, get a locally analytic P-rep

$$\mathrm{Lift}(M, \log)$$

which induces original $\mathfrak{p}$ action, where log is a logarithm on T.

Now, for general M, use a left Kan extension:

still locally finite / $\mathfrak{p}$ and $\mathfrak{u}$ acts nilpotently.

$$\begin{tikzcd}
\text{fd-ones} \arrow[r, hook] \arrow[dr, bend right, "\mathrm{Lift}(-,\log)"'] & \text{all} \arrow[d, dashed] \\
& \mathrm{Rep}_P^{an}
\end{tikzcd}$$

Left Kan extension fills in this diagram in a universal way

Concretely, $\mathrm{Lift}(M, \log) = \operatorname*{colim}_{\mathrm{fd}\, N \subseteq M} \mathrm{Lift}(N, \log)$

If P is parabolic in a split reductive G and $M \in \mathcal{O}^{\mathfrak{p},\infty}_{\mathrm{alg}}$, have an action of P on $\mathrm{Lift}(M, \log)$ which makes this a $D(\mathfrak{g}, P)$-module. [AS, 3.3]

If V is a smooth strongly admissible P-rep, then V' is a $D(P)$-module on which $\mathfrak{p}$ acts trivially, so it's also a $D(\mathfrak{g}, P)$-module. [AS, 4.1.3]

Next: want $\mathrm{Lift}(M, \log) \otimes V'$ to be a $D(\mathfrak{g}, P)$-module.

④ Tensor-Hom adjunction for $D(H)$ and $D(\mathfrak{g},H)$

Let G be a locally analytic group and $H \subseteq G$ a closed subgroup.

$D(H)$ is not a Hopf algebra: there isn't a comultiplication
$$D(H) \longrightarrow D(H) \otimes D(H)$$

This means that the tensor product (over F) of two $D(H)$-modules isn't automatically a $D(H)$-module.

But there is a "comultiplication" [Schneider-Teitelbaum '05]
$$D(H) \longrightarrow D(H) \hat{\otimes} D(H)$$

which is good enough to make the following work:

[AS, 6.3.1]

Thm. Let M be a locally finite dimensional locally analytic representation of H.
For any $D(H)$-modules X, Y, ← no topology!
$$M \otimes X \quad \& \quad \mathrm{Hom}(M, Y)$$
are naturally $D(H)$-modules and
$$\mathrm{Hom}_{D(H)}(M \otimes X, Y) = \mathrm{Hom}_{D(H)}(X, \mathrm{Hom}(M,Y))$$

Then, since $U(\mathfrak{g})$ is a Hopf algebra, we can bootstrap up from the above adjunction to get:

[AS, 7.4.5]

Thm. Let M be a locally finite dimensional locally analytic representation of H which is also compatibly a $\mathfrak{g}$-module.
For any $D(\mathfrak{g},H)$-modules X, Y,
$$M \otimes X \quad \& \quad \mathrm{Hom}(M, Y)$$
are naturally $D(\mathfrak{g},H)$-modules and
$$\mathrm{Hom}_{D(\mathfrak{g},H)}(M \otimes X, Y) = \mathrm{Hom}_{D(\mathfrak{g},H)}(X, \mathrm{Hom}(M,Y))$$

In particular, if G split reductive and P parabolic and $M \in \mathcal{O}^{\mathfrak{p},\infty}_{\mathrm{alg}}$, then
$$\mathrm{Lift}(M, \log) = \operatorname*{colim}_{\text{fd } \mathfrak{p}\text{-submods } N} \mathrm{Lift}(N, \log)$$
is locally finite dimensional as a representation of P.
$\Rightarrow$ $\mathrm{Lift}(M,\log) \otimes V'$ is a $D(\mathfrak{g}, P)$-module.

⑤ Functor and its properties

G split reductive, T split maximal torus, P parabolic containing T, $\log \in \mathrm{Logs}(T)$, G_0 maximal compact, and $P_0 = P \cap G_0$.

It's important here that V is strongly admissible and not just admissible

[AS, 4.1.5] **Proposition.** If $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p},\infty}$ and V is a smooth strongly admissible representation of P, then

$$\mathrm{Lift}(M, \log) \otimes V'$$

is a finitely presented $D(\mathfrak{g}, P_0)$-module.

Define

$$\check{\mathcal{F}}_P^G(M,V) = D(G) \otimes_{D(\mathfrak{g},P)} \left(\mathrm{Lift}(M,\log)\otimes V'\right)$$

as $D(G_0)$-mods

$$= D(G_0) \otimes_{D(\mathfrak{g},P_0)} \left(\mathrm{Lift}(M,\log)\otimes V'\right)$$

[AS, 4.2.3] **Thm.** $\check{\mathcal{F}}_P^G(M,V)$ is coadmissible.

Proof. $\mathrm{Lift}(M,\log)\otimes V'$ is finitely presented over $D(\mathfrak{g},P_0)$, so $\check{\mathcal{F}}_P^G(M,V)$ is finitely presented over $D(G_0)$ □

[AS, 4.2.4] **Thm.** $\check{\mathcal{F}}_P^G$ is exact in both variables.

Proof sketch. Break up $D(\mathfrak{g},P_0) \longrightarrow D(G_0)$ into two "steps":

$$\begin{tikzcd}
D(\mathfrak{g},P_0) \arrow[rr] \arrow[dr, "\text{``algebraic'' (finite presentation)}"'] & & D(G_0) \\
& D(G_0)_{P_0} \arrow[ur, "\text{``topological'' (c-flatness)}"'] &
\end{tikzcd}$$
□

Kohlhaase's supported distributions = closure of $D(\mathfrak{g},P_0)$ inside $D(G_0)$.

[AS, 4.3.3] **Thm.** Suppose $P \subseteq Q \subseteq G$. If $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{q},\infty}$ and V is a smooth strongly admissible representation of P, then

$$\check{\mathcal{F}}_P^G(M,V) = \check{\mathcal{F}}_Q^G(M, i(V)).$$

$\mathcal{O}_{\text{alg}}^{\mathfrak{p},\infty} \supseteq \mathcal{O}_{\text{alg}}^{\mathfrak{q},\infty}$

smooth induction $\mathrm{ind}_P^Q(V)$

Proof sketch. Use tensor-hom adjunction for $D(\mathfrak{g},P_0)$ plus the fact that

$$i(V)' = D(\mathfrak{g},Q) \otimes_{D(\mathfrak{g},P)} V' \quad \text{as } D(\mathfrak{g},Q)\text{-modules.}$$
□

Thanks! :)

Questions?