

From category \mathcal{O}^∞ to locally analytic representations

Joint work with Matthias Strauch

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- ⑤ Functor and its properties

F/\mathbb{Q}_p a finite extension

① Locally analytic representations

Let G be a locally analytic group / F

Ex. if G is an algebraic group / F , then $G := G(F)$ is a locally analytic group.

Closed subgroups of this (in the p -adic topology) are also locally analytic groups.

A locally analytic representation of G is a 'reasonable' topological vector space V over F with a continuous action of G such that, for any $v \in V$, the orbit map

$$o_v: G \rightarrow V$$

is a locally analytic function on G .

barrelled locally convex vector space

$G \times V \rightarrow V$ is continuous

better: finite extn E/F for coefficients

Examples:

- Any smooth representation (o_v is locally constant)
- G reductive and V is finite dim'l algebraic rep
- $G = \mathrm{GL}_2(F)$ and V is 1-forms on Drinfeld's " p -adic upper half plane" ($= \mathbb{P}^1 \setminus \mathbb{P}^1(F)$)
- (*) • $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and V is the subspace of locally analytic vectors in the Banach space representation attached to a 2-dimensional p -adic Galois representation via Colmez's p -adic Langlands correspondence
- Etc.

Montréal functor

{ admissible unitary Banach reps of $\mathrm{GL}_2(\mathbb{Q}_p)$ over F which have a central character and which are residually of finite length }

{ 2-dim'l reps of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ over F }

which, roughly, induces bijection between iso classes of absolutely irreducible objects

Goal: Produce a class of locally analytic representations that can be understood using Lie algebra representations

[Turns out that the class we'll produce is related to (*) for semistable non-crystalline Galois reps]

Breuil constructs a Banach rep $B(k, \lambda)$ corresponding

Algebraic perspective on locally analytic reps:

$D(G) = C(G)$ is distributions on G

↖ continuous dual of locally analytic functions on G

This is naturally a topological F -algebra

↳ group algebra

• $F[G]$ is a dense subalgebra of $D(G)$

$$g \mapsto [\delta_g : f \mapsto f(g)] \in D(G)$$

↖ not commutative, not noetherian, is Fréchet-Stein if G is compact

• If $\mathfrak{g} = \text{Lie}(G)$, then $U(\mathfrak{g})$ is also a subalgebra of $D(G)$

$$x \cdot f = \lim_{t \rightarrow 0} \frac{\exp(tx)f - f}{t}$$

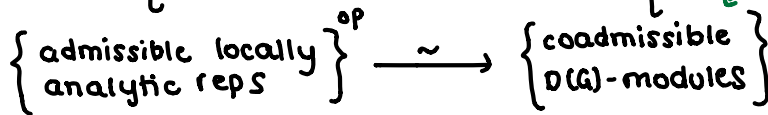
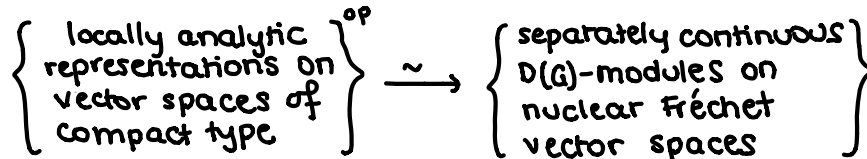
$$x \mapsto [f \mapsto ((-x) \cdot f)(1)] \in D(G)$$

If V is a locally analytic representation, then

$$\delta_g \cdot v = g \cdot v$$

extends to a separately continuous $D(G)$ -module structure on V .

Thm (Schneider-Teitelbaum)



↖ a kind of finiteness condition (eg, finitely presented \Rightarrow coadmissible when G is compact)

↳ (not closed)

If $H \subseteq G$ is a closed subgroup, can consider subring $D(\mathfrak{g}, H)$ generated by $U(\mathfrak{g})$ and $D(H)$.

Locally analytic reps of H that are "compatibly" \mathfrak{g} -modules are naturally $D(\mathfrak{g}, H)$ -modules.

↖ Two actions of $\mathfrak{h} = \text{Lie}(H)$ agree, ξ
 $\delta_h(xm) = \text{Ad}(h)(x)(h.m)$
 for $h \in H, x \in \mathfrak{g}, m \in M$.

More about all of this :

Schneider-Teitelbaum '01, '02, '03, '05

Kohlhaase '07

Schmidt-Strauch '16

Emerton '17

[AS] sections 6 & 7

⋮

② Outline of the construction

Notation

• G a split reductive group / F

• T a split maximal torus

to semistable non-cryst. Galois reps. It's a completion of a locally an rep $\Sigma(k, \mathbb{Z})$, which in turn is a subquotient of $\text{Ind}_P^G(\sigma(k, \mathbb{Z}))$, which is in the image of our functor.

When $k=2 \dots$

Let N be \mathfrak{g} -module

$$\begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} 0 & x-y \\ 0 & 0 \end{pmatrix}$$

and $M = \text{Ind}_{\mathbb{Z}}^{\mathfrak{g}}(N)$. Then

$$\text{Ind}_P^G(\sigma(k, \mathbb{Z})) = \mathfrak{S}_P^G(M, \mathbb{Z})$$

where \mathbb{Z} on left corresp. to choice of log on right.

- P a parabolic containing T
- $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{t} = \text{Lie}(T)$, $\mathfrak{p} = \text{Lie}(P)$

BGG category $\mathcal{O}^{\mathbb{F}}$

- (1) Finitely generated as \mathfrak{g} -module
- (2) locally finite dimensional as \mathfrak{p} -module
- (3) weights defined over F
- (4) Action of \mathfrak{t} is semisimple \leftarrow not stable under extensions!

Extension closure of $\mathcal{O}^{\mathbb{F}}$ is "thick" BGG category $\mathcal{O}^{\mathbb{F}, \infty}$
 defined by conditions (1), (2), & (3) [No (4)]

Write $\mathcal{O}_{\text{alg}}^{\mathbb{F}}$, $\mathcal{O}_{\text{alg}}^{\mathbb{F}, \infty}$ for subcategories of \mathfrak{g} -modules
 which have algebraic weights

Thm. There is a bifunctor

[AS, 4.2.3-4]

$$\mathcal{O}_{\text{alg}}^{\mathbb{F}, \infty} \times \left\{ \begin{array}{l} \text{smooth strongly} \\ \text{admissible reps} \\ \text{of } P \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{admissible locally} \\ \text{analytic reps} \\ \text{of } G \end{array} \right\}$$

$$(M, V) \longmapsto \mathcal{F}_P^G(M, V)$$

which is contravariant in M , covariant in V , and exact in both.

Orlik-Strauch '14: $\mathcal{O}_{\text{alg}}^{\mathbb{F}}$ in the first entry, any admissible smooth rep of P in the second.

Why extend from $\mathcal{O}^{\mathbb{F}}$ to $\mathcal{O}^{\mathbb{F}, \infty}$?

- $\mathcal{O}^{\mathbb{F}}$ not being stable under extensions can be technically inconvenient
- \mathfrak{g} -modules that are not in category $\mathcal{O}^{\mathbb{F}}$ do come up!

Ex. $G = \text{SL}_2$ $P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ $X = G/P = \mathbb{P}^1$

There are five indecomposable D -mods on X

$$D/\partial \quad D/\partial \quad D/\partial \partial \quad D/\partial \partial \quad D/\partial \partial \partial$$

Taking $\Gamma(X, -)$ is an equivalence \leftarrow Beilinson-Bernstein localization
 $D\text{-Mod} \longrightarrow \mathfrak{g}\text{-Mod}_0$

and

$$\Gamma(X, D/\partial \partial \partial) \in \mathcal{O}^{\infty} \setminus \mathcal{O}$$

Ex. Representations of form $\mathcal{F}_P^G(M, V)$ w/ $M \in \mathcal{O}^{\infty} \setminus \mathcal{O}$ are related to reps that occur in work of Breuil & Schraen in the p -adic Langlands program. [AS, section 5]

To construct $\mathcal{F}_P^G(M, V) \dots$

choice of logarithm doesn't matter for $M \in \mathcal{O}^{\mathbb{F}}$

(a) Use a p-adic logarithm to construct a locally analytic action of P on M lifting action of \mathfrak{p} .
 M with this action of P is $\text{Lift}(M, \log)$
 This is a $D(\mathfrak{g}, P)$ -module

(b) V' is a $D(P)$ -module on which \mathfrak{p} acts trivially, so can regard it is a $D(\mathfrak{g}, P)$ -module with a trivial action of \mathfrak{g} .

(c) $\text{Lift}(M, \log) \otimes V'$ is a $D(\mathfrak{g}, P)$ -module

even though $D(\mathfrak{g}, P)$ is not a Hopf algebra cf (4)

(d) Define

$$\check{\mathcal{F}}_P^G(M, V) = D(\mathfrak{g}) \otimes_{D(\mathfrak{g}, P)} [\text{Lift}(M, \log) \otimes V']$$

and

$$\mathcal{F}_P^G(M, V) = [\check{\mathcal{F}}_P^G(M, V)]'$$

③ Lifting Lie algebra representations [AS, section 2]

Let P be a connected algebraic group / F with split maximal torus T and unipotent radical U .

Let $\mathfrak{p} = \text{Lie}(P)$, $\mathfrak{t} = \text{Lie}(T)$, $\mathfrak{u} = \text{Lie}(U)$.

Let M be a \mathfrak{p} -module such that

- Locally finite dimensional as \mathfrak{p} -module
- locally nilpotent action of \mathfrak{u}
- Algebraic weights

Eg, the \mathfrak{p} -module underlying any object of $\mathcal{O}_{\text{alg}}^{\mathfrak{p}, \infty}$

We'll construct an action of P on M which "lifts" the original action of \mathfrak{p} ↑ differentiates to

Until further notice, M is finite dimensional.

We'll gradually increase the "complexity" of P .

(i) If $P=U$, we have group isomorphisms

$$U \begin{matrix} \xrightarrow{\log} \\ \xleftarrow{\exp} \end{matrix} \mathfrak{u}$$

Set of nilpotent linear maps $M \rightarrow M$

Since the action of \mathfrak{u} on M is nilpotent, we can exponentiate it, and

$$U \xrightarrow{\log} \mathfrak{u} \longrightarrow \text{Nil}(M) \xrightarrow{\exp} \text{GL}(M)$$

is an action of U lifting the original action of \mathfrak{u} .

important case!

(ii) Suppose $P=T$.

Since we're over a p-adic field, have $\exp: t \rightarrow T$
 but there exist homomorphisms $T \rightarrow t$ which invert \exp .
 These are logarithms on T .

Ex. $T = G_m$ $F = \mathbb{Q}_p$

Maps $\mathbb{Q}_p^\times \rightarrow \mathbb{Q}_p$ inverting \exp are
 determined by where p goes.

Choosing $p \mapsto 0$ gives the "Iwasawa log."

If t acts nilpotently on M , can now use a log on T
 to construct action of T on M just as before:

$$T \xrightarrow{\log} t \longrightarrow \text{Nil}(M) \xrightarrow{\exp} GL(M)$$

Ex. $T = G_m$

2 dim'l \curvearrowright $M \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ \curvearrowleft action of $t \in t$

Then can lift the action of $t = F$ to an action
 of $T = F^\times$ using the map $\rho: F^\times \rightarrow GL_2(F)$

$$\rho(a) = \begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}$$

[AS.2.3.2-3] That deals with the case where 0 is the only
 weight. For other weights, we "twist", and if
 there's more than one weight, we just lift one
 generalized weight space at a time.

Ex. $T = G_m$

$$M \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

\curvearrowleft all weights are
 still assumed to
 be algebraic!

can lift using $\rho: F^\times \rightarrow GL_2$ given by

$$\rho(a) = a^2 \begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}$$

\curvearrowright character of T
 that differentiates
 to the weight 2

\curvearrowleft what we had
 above

Let $\text{Lift}(M, \log)$ be this locally analytic rep'n of T .

Note that the log comes up in dealing with
 non-semisimplicity. If the Lie algebra rep is
 semisimple (ie, generalized weight spaces are
 just weight spaces), the log doesn't matter.

\curvearrowright eg, the t -module
 underlying an obj
 in category $\mathcal{O}^?$!

(iii) \mathfrak{p} semisimple

Jantzen does this in "Representations of alg. groups" §7.
 Basically, idea is just to glue actions of root

[AS, 2.4.8]

subgroups (which are unipotent) with action of torus.

Since P semisimple, action of t is forced to be semisimple, so the lifted action is even algebraic (no logs show up!)

(iv) P reductive

[AS, 2.4.9]

Let $P' = [P, P]$ be derived subgroup. Get an action of P' using (iii). Have action of T from (ii) if we choose a logarithm. Actions are compatible, and P is generated by T & P' , so done.

agree on $T \cap P'$, and compatible w/ conjugation $T \cap P'$

(v). P general

[AS, 2.4.11]

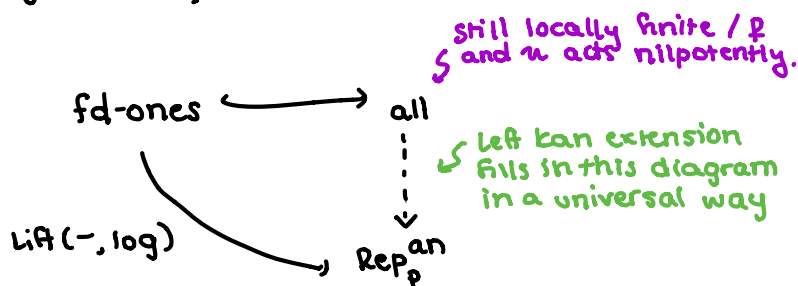
Let L be a Levi subgroup so that $P = U \rtimes L$. L is reductive, so get action using (iv). U unipotent, so get an action using (i). Again compatible, so get an action of P .

Upshot: For any finite dimensional \mathfrak{p} -module M on which u acts nilpotently, get a locally analytic P -rep

$$\text{Lift}(M, \log)$$

which induces original \mathfrak{g} action, where \log is a logarithm on T .

Now, for general M , use a left Kan extension:



$$\text{Concretely, } \text{Lift}(M, \log) = \text{colim}_{\text{fd } N \subseteq M} \text{Lift}(N, \log)$$

If P is parabolic in a split reductive G and $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}, \infty}$, have an action of P on $\text{Lift}(M, \log)$ which makes this a $D(\mathfrak{g}, P)$ -module. [AS, 3.3]

If V is a smooth strongly admissible P -rep, then V' is a $D(P)$ -module on which \mathfrak{g} acts trivially, so it's also a $D(\mathfrak{g}, P)$ -module. [AS, 4.1.3]

Next: want $\text{Lift}(M, \log) \otimes V'$ to be a $D(\mathfrak{g}, P)$ -module.

(4) Tensor-Hom adjunction for $D(H)$ and $D(\mathfrak{g}, H)$

Let G be a locally analytic group and $H \subseteq G$ a closed subgroup.

$D(H)$ is not a Hopf algebra: there isn't a comultiplication

$$D(H) \longrightarrow D(H) \otimes D(H)$$

This means that the tensor product (over F) of two $D(H)$ -modules isn't automatically a $D(H)$ -module.

But there is a "comultiplication" [Schneider-Teitelbaum '05]

$$D(H) \longrightarrow D(H) \hat{\otimes} D(H)$$

which is good enough to make the following work:

Thm. Let M be a locally finite dimensional locally analytic representation of H .

[AS, 6.3.1] For any $D(H)$ -modules X, Y , \longleftarrow no topology!

$$M \otimes X \quad \doteq \quad \text{Hom}(M, Y)$$

are naturally $D(H)$ -modules and

$$\text{Hom}_{D(H)}(M \otimes X, Y) = \text{Hom}_{D(H)}(X, \text{Hom}(M, Y))$$

Then, since $U(\mathfrak{g})$ is a Hopf algebra, we can bootstrap up from the above adjunction to get:

[AS, 7.4.5] Thm. Let M be a locally finite dimensional locally analytic representation of H which is also compatibly a \mathfrak{g} -module.

For any $D(\mathfrak{g}, H)$ -modules X, Y ,

$$M \otimes X \quad \doteq \quad \text{Hom}(M, Y)$$

are naturally $D(\mathfrak{g}, H)$ -modules and

$$\text{Hom}_{D(\mathfrak{g}, H)}(M \otimes X, Y) = \text{Hom}_{D(\mathfrak{g}, H)}(X, \text{Hom}(M, Y))$$

In particular, if G split reductive and P parabolic and $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}, \infty}$, then

$$\text{Lift}(M, \log) = \text{colim}_{\text{Ad } P\text{-submods } N} \text{Lift}(N, \log)$$

is locally finite dimensional as a representation of P .

$\Rightarrow \text{Lift}(M, \log) \otimes V^1$ is a $D(\mathfrak{g}, P)$ -module.

(5) Functor and its properties

G split reductive, T split maximal torus, P parabolic containing T , $\log \in \text{Log}(T)$, G_0 maximal compact, and $P_0 = P \cap G_0$.

It's important here that V is strongly admissible and not just admissible

[AS, 4.1.5]

Proposition. If $M \in \mathcal{O}_{\text{alg}}^{\text{f}, \infty}$ and V is a smooth strongly admissible representation of P , then

$$\text{Lift}(M, \log) \otimes V'$$

is a finitely presented $D(\mathfrak{g}, \mathfrak{p}_0)$ -module.

define

$$\check{F}_P^G(M, V) = D(G) \otimes_{D(\mathfrak{g}, P)} (\text{Lift}(M, \log) \otimes V')$$

as $D(G_0)$ -mods

$$= D(G_0) \otimes_{D(\mathfrak{g}, \mathfrak{p}_0)} (\text{Lift}(M, \log) \otimes V')$$

[AS, 4.2.3]

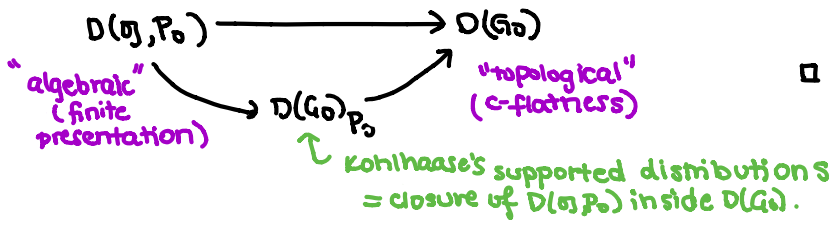
Thm. $\check{F}_P^G(M, V)$ is coadmissible.

Proof. $\text{Lift}(M, \log) \otimes V'$ is finitely presented over $D(\mathfrak{g}, \mathfrak{p}_0)$, so $\check{F}_P^G(M, V)$ is finitely presented over $D(G_0)$ \square

[AS, 4.2.4]

Thm. \check{F}_P^G is exact in both variables.

Proof sketch. Break up $D(\mathfrak{g}, \mathfrak{p}_0) \rightarrow D(G_0)$ into two "steps":



[AS, 4.3.3]

Thm. Suppose $P \subseteq Q \subseteq G$. If $M \in \mathcal{O}_{\text{alg}}^{\text{f}, \infty}$ and V is a smooth strongly admissible representation of P , then

$$\check{F}_P^G(M, V) = \check{F}_Q^G(M, i(V)).$$

$\mathcal{O}_{\text{alg}}^{\text{f}, \infty} \cong \mathcal{O}_{\text{alg}}^{\text{f}, \infty}$ smooth induction $\text{ind}_P^Q(V)$

Proof sketch. Use tensor-hom adjunction for $D(\mathfrak{g}, \mathfrak{p}_0)$ plus the fact that

$$i(V)' = D(\mathfrak{g}, \mathfrak{q}) \otimes_{D(\mathfrak{g}, P)} V' \text{ as } D(\mathfrak{g}, \mathfrak{q})\text{-modules. } \square$$

Thanks! 😊

Questions?