From category $\mathcal{O}^\infty$ to locally analytic representations

Joint work with Matthias Strauch
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Padova, p-adic representations & arithmetic D-modules

Outline
1. Introduction
2. Lifting Lie algebra representations
3. Tensor-hom adjunction for $D(H)$ & $D(\mathfrak{g}, H)$
4. The functor and its properties

1 Introduction

Notation:
- $F/\mathbb{Q}_p$ a finite extension
- (G,T) a split reductive group
- P a parabolic containing T
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{p} = \mathrm{Lie}(P)$

Recall:
$\mathfrak{g}$-module $M \in \mathcal{O}^{\mathfrak{p}}$ if:
(1) Finitely generated as $\mathfrak{g}$-module
(2) Locally finite dimensional as $\mathfrak{p}$-module
(3) Weights defined over F
(4) Action of $\mathfrak{t}$ is semisimple

better: finite ext'n E/F for coefficients

not stable under extensions!

If weights of M all come from algebraic characters of T, then $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$.

There is a bifunctor

$$(M, V) \longmapsto \mathcal{F}^G_P(M, V)$$

M: $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$

V: (strongly) admissible smooth representation of P

$\mathcal{F}^G_P(M, V)$: admissible representation of G

which is contravariant in M, covariant in V, and exact in both M & V. [Orlik-Strauch, 2014]

To construct $\mathcal{F}^G_P(M,V)$...
(a) There's a natural action of P on M which is compatible with the action of $\mathfrak{g}$, ie, M is a $D(\mathfrak{g}, P)$-module

Recall: $D(\mathfrak{g}, P)$ is subalgebra of $D(G)$ generated by $U(\mathfrak{g})$ and $D(P)$

strong dual

(b) V' is a $D(P)$-module where $\mathfrak{p}$ acts trivially, so regard as $D(\mathfrak{g}, P)$-module w/ trivial $\mathfrak{g}$-action
(c) $M \otimes V'$ is a $D(\mathfrak{g}, P)$-module

even though

(d) Define
$$\check{\mathcal{F}}^G_P(M,V) = D(G) \otimes_{D(\mathfrak{g},P)} [M \otimes V']$$
and then
$$\mathcal{F}^G_P(M,V) = \left[\check{\mathcal{F}}^G_P(M,V)\right]'$$

$D(\mathfrak{g},P)$ is not a Hopf algebra! cf. section ③

Want to extend the domain of $\mathcal{F}^G_P$ from $M \in \mathcal{O}^{\mathfrak{p}}_{alg}$ to $M \in \mathcal{O}^{\mathfrak{p},\infty}_{alg}$ — the extension closure of $\mathcal{O}^{\mathfrak{p}}_{alg}$ in the category of $\mathfrak{g}$-modules.

[$\mathfrak{g}$-module $M \in \mathcal{O}^{\mathfrak{p},\infty}$ if:
(1) Finitely generated as $\mathfrak{g}$-module
(2) Locally finite dimensional as $\mathfrak{p}$-module
(3) Weights defined over F
[No condition (4) !]]

There's no "natural" action of P on M anymore, but if we choose a logarithm, we can construct a functorial action of P on M. Cf section ②

Why do we want to do this?

- $\mathcal{O}^{\mathfrak{p}}$ not being stable under extensions can be technically inconvenient
- $\mathfrak{g}$-modules that are not in category $\mathcal{O}^{\mathfrak{p}}$ do come up!

Ex. $G = SL_2$ $P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ $X = G/P = \mathbb{P}^1$
There are five indecomposable D-mods on X
D/∂ D/t $D/\partial t$ $D/t\partial$ $D/t\partial t$

Taking $\Gamma(X,-)$ is an equivalence (Beilinson-Bernstein localization)
$$D\text{-Mod} \longrightarrow \mathfrak{g}\text{-Mod}_0$$
and
$$\Gamma(X, D/t\partial t) \in \mathcal{O}^{\infty} \setminus \mathcal{O} \quad !$$

Ex. Representations of form $\mathcal{F}^G_P(M,V)$ w/ $M \in \mathcal{O}^{\infty} \setminus \mathcal{O}$ are related to reps that occur in work of Breuil & Schraen in the p-adic Langlands program. [AS, section 5]

$\mathcal{O}^{\mathfrak{p},\infty}_{alg}$ is a Serre subcategory of $\mathfrak{g}$-Mod, so can form $D_{\mathcal{O}^{\mathfrak{p},\infty}_{alg}}(\mathfrak{g}\text{-Mod})$, ie, derived cat. of $\mathfrak{g}$-Mod with cohomology in $\mathcal{O}^{\mathfrak{p},\infty}_{alg}$.

$$\rightsquigarrow \quad D_{\mathcal{O}^{\mathfrak{p},\infty}_{alg}}(\mathfrak{g}\text{-Mod}) \xrightarrow{\check{\mathcal{F}}^G_P} D_{coadm}(D(G))$$

② Lifting Lie Algebra Representations

[AS, section 2]

Let P be a connected algebraic group $/F$ with split maximal torus T and unipotent radical U.
Let $\mathfrak{p} = \mathrm{Lie}(P)$, $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{u} = \mathrm{Lie}(U)$.

Let M be a $\mathfrak{p}$-module such that

Thanks, Simon!

Eg. the $\mathfrak{p}$-module

- Locally finite dimensional as $\mathfrak{p}$-module
- locally nilpotent action of $\mathfrak{u}$
- Algebraic weights

eg, the $\mathfrak{p}$-module underlying any object of $\mathcal{O}^{\mathfrak{p},\infty}_{alg}$

differentiates to

We'll construct an action of P on M which "lifts" the original action of $\mathfrak{p}$

Until further notice, M is finite dimensional.

We'll gradually increase the "complexity" of P.

(i) If $P = U$, we have group isomorphisms

$$U \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{u}$$

Since the action of $\mathfrak{u}$ on M is nilpotent, we can exponentiate it, and

set of nilpotent linear maps $M \to M$

important case!

$$U \xrightarrow{\log} \mathfrak{u} \longrightarrow \mathrm{Nil}(M) \xrightarrow{\exp} GL(M)$$

is an action of U lifting the original action of $\mathfrak{u}$.

(ii) Suppose $P = T$.

We have an exponential map $\exp : \mathfrak{t} \dashrightarrow T$. Since we're over a p-adic field, there exist homomorphisms $T \to \mathfrak{t}$ which invert $\exp$! Let $\mathrm{Logs}(T)$ be the set of these logarithms.

Ex. $T = \mathbb{G}_m$ $F = \mathbb{Q}_p$
Maps $\mathbb{Q}_p^\times \longrightarrow \mathbb{Q}_p$ inverting $\exp$ are determined by where p goes.
Choosing $p \mapsto 0$ gives the "Iwasawa log."

If $\mathfrak{t}$ acts nilpotently on M, can now use a $\log \in \mathrm{Logs}(T)$ to construct action of T on M just as before:

$$T \xrightarrow{\log} \mathfrak{t} \longrightarrow \mathrm{Nil}(M) \xrightarrow{\exp} GL(M)$$

Ex. $T = \mathbb{G}_m$

2 dim'l   action of let

$$M \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then can lift the action of $\mathfrak{t} = F$ to an action of $T = F^\times$ using the map $\rho : F^\times \longrightarrow GL_2(F)$

$$\rho(a) = \begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}$$

[3.2–3] That deals with the case where 0 is the only weight. For other weights, we "twist" and if

[AS, 2.3] there's more than one weight, we just lift one generalized weight space at a time.

all weights are still assumed to be algebraic!

Ex. $T = \mathbb{G}_m$

$$M \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Can lift using $\rho : F^\times \longrightarrow GL_2$ given by

$$\rho(a) = a^2 \begin{bmatrix} 1 & \log(a) \\ 0 & 1 \end{bmatrix}$$

character of T that differentiates to the weight 2

what we had above

Let $\mathrm{Lift}(M, \log)$ be this locally analytic rep'n of T.

Note that the log comes up in dealing with non-semisimplicity. If the Lie algebra rep is semisimple (ie, generalized weight spaces are just weight spaces), the log doesn't matter.

eg, the t-module underlying an obj in category $\mathcal{O}$!

(iii) P semisimple

Jantzen does this in "Representations of alg. groups." Basically, idea is just to glue actions of root subgroups (which are unipotent) with action of torus.

[AS, 2.4.8] Since P semisimple, action of $\mathfrak{t}$ is forced to be semisimple, so the lifted action is even algebraic (no logs show up!)

(iv) P reductive

[AS, 2.4.9] Let $P' = [P,P]$ be derived subgroup. Get an action of P' using (iii). Have action of T from (ii) if we choose a logarithm. Actions are compatible, and P is generated by T & P', so done.

agree on $T \cap P'$, and compatible w/ conjugation $T \curvearrowright P'$

(v). P general

[AS, 2.4.11] Let L be a Levi subgroup so that $P = U \rtimes L$. L is reductive, so get action using (iv). U unipotent, so get an action using (i). Again compatible, so get an action of P.

Upshot: For any finite dimensional $\mathfrak{p}$-module M on which $\mathfrak{u}$ acts nilpotently, get a locally analytic P-rep

$$\mathrm{Lift}(M, \log),$$

where $\log \in \mathrm{Logs}(T)$, which induces original $\mathfrak{p}$-module M.

Now, for general M, use a left Kan extension:

$$\begin{tikzcd} \text{fd-ones} \arrow[r, hook] \arrow[dr, bend right, "\mathrm{Lift}(-,\log)"'] & \text{all} \arrow[d, dashed] \\ & \mathrm{Rep}_{\mathfrak{p}}^{an} \end{tikzcd}$$

still locally finite / $\mathfrak{p}$ and $\mathfrak{u}$ acts nilpotently.

left kan extension fills in this diagram in a universal way

Concretely, $\mathrm{Lift}(M,\log) = \operatorname*{colim}_{\text{fd } N \subseteq M} \mathrm{Lift}(N,\log)$

If P is parabolic in a split reductive G and $M \in \mathcal{O}^{\mathfrak{p},\infty}_{\text{alg}}$, have an action of P on $\mathrm{Lift}(M,\log)$ which makes this a $D(\mathfrak{g},P)$-module.

Next: If V is a smooth strongly admissible P-rep, then V' is a $D(P)$-module on which $\mathfrak{p}$ acts trivially, so it's also a $D(\mathfrak{g},P)$-module. [AS, section 3.3]

want $\mathrm{Lift}(M,\log) \otimes V'$ to be a $D(\mathfrak{g},P)$-module.

(3) Tensor-Hom Adjunction for $D(H)$ & $D(\mathfrak{g},H)$

Let G be a locally analytic group and $H \subseteq G$ a closed subgroup.

$D(H)$ is not a Hopf algebra: there isn't a comultiplication

$$D(H) \longrightarrow D(H) \otimes D(H)$$

This means that the tensor product (over F) of two $D(H)$-modules isn't automatically a $D(H)$-module.

But there is a "comultiplication" [Schneider-Teitelbaum, 05]

$$D(H) \longrightarrow D(H) \hat{\otimes} D(H)$$

which is good enough to make the following work:

[AS, 6.3.1]

Thm. Let M be a locally finite dimensional locally analytic representation of H.

can be regarded as a $D(H)$-module where $\delta_h \cdot m = h \cdot m$

For any $D(H)$-modules X, Y,

no topology!

$$M \otimes X \quad \& \quad \mathrm{Hom}(M,Y)$$

are naturally $D(H)$-modules and

$$\mathrm{Hom}_{D(H)}(M \otimes X, Y) = \mathrm{Hom}_{D(H)}(X, \mathrm{Hom}(M,Y))$$

Then, since $U(\mathfrak{g})$ is a Hopf algebra, we can bootstrap up from the above adjunction to get:

[AS, 7.4.5]

Thm. Let M be a locally finite dimensional locally analytic representation of H which is also compatibly a $\mathfrak{g}$-module.

Two actions of $\mathfrak{h} = \mathrm{Lie}(H)$ agree, and
$\delta_h(xm) = \mathrm{Ad}(h)(x)(h.m)$
for $h \in H$, $x \in \mathfrak{g}$, $m \in M$.
$\rightsquigarrow$ M is a $D(\mathfrak{g},H)$-module.

For any $D(\mathfrak{g},H)$-modules X, Y,
$$M \otimes X \quad \& \quad \mathrm{Hom}(M,Y)$$
are naturally $D(\mathfrak{g},H)$-modules and
$$\mathrm{Hom}_{D(\mathfrak{g},H)}(M \otimes X, Y) = \mathrm{Hom}_{D(\mathfrak{g},H)}(X, \mathrm{Hom}(M,Y))$$

In particular, if G split reductive and P parabolic and $M \in \mathcal{O}_{alg}^{\mathfrak{p},\infty}$, then
$$\mathrm{Lift}(M,\log) = \operatorname*{colim}_{\text{fd } \mathfrak{p}\text{-submods } N} \mathrm{Lift}(N,\log)$$
is locally finite dimensional as a representation of P.
$\Rightarrow$ $\mathrm{Lift}(M,\log) \otimes V'$ is a $D(\mathfrak{g},P)$-module.

(4) The Functor and its Properties

G split reductive, T split maximal torus, P parabolic containing T, $\log \in \mathrm{Logs}(T)$, G_0 maximal compact, and $P_0 = P \cap G_0$.

[AS, 4.1.5]

Proposition. If $M \in \mathcal{O}_{alg}^{\mathfrak{p},\infty}$ and V is a smooth strongly admissible representation of P, then
$$\mathrm{Lift}(M,\log) \otimes V'$$
is a finitely presented $D(\mathfrak{g},P_0)$-module.

It's important here that V is strongly admissible and not just admissible

Define
$$\check{\mathcal{F}}_P^G(M,V) = D(G) \otimes_{D(\mathfrak{g},P)} \left(\mathrm{Lift}(M,\log) \otimes V'\right)$$

as $D(G_0)$-mods
$$= D(G_0) \otimes_{D(\mathfrak{g},P_0)} \left(\mathrm{Lift}(M,\log) \otimes V'\right)$$

[AS, 4.2.3–4]

Thm. $\check{\mathcal{F}}_P^G(M,V)$ is coadmissible.

Proof. $\mathrm{Lift}(M,\log) \otimes V'$ is finitely presented over $D(\mathfrak{g},P_0)$, so $\check{\mathcal{F}}_P^G(M,V)$ is finitely presented over $D(G_0)$ $\square$

Thm. $\check{\mathcal{F}}_P^G$ is exact in both variables.

Same proof that Matthias discussed!

$$\begin{tikzcd}
D(\mathfrak{g},P_0) \arrow[rr] \arrow[dr, bend right] & & D(G_0) \\
& D(G_0)_{P_0} \arrow[ur, bend right] &
\end{tikzcd}$$

"algebraic" (use finite presentation)

"topological" (use c-flatness)

Kohlhaase's supported distributions

$=$ closure of $D(\mathfrak{g}, P_0)$ inside $D(G_0)$.

[AS, 4.3.3]

Thm. Suppose $P \subseteq Q \subseteq G$. If $M \in \mathcal{O}_{alg}^{q,\infty}$ and V is a smooth strongly admissible representation of P, then

$$\check{\mathcal{F}}_P^G(M, V) = \check{\mathcal{F}}_Q^G(M, i(V)).$$

$\mathcal{O}_{alg}^{p,\infty} \supseteq \mathcal{O}_{alg}^{q,\infty}$

smooth induction $\mathrm{ind}_P^Q(V)$

Proof. Let $\tilde{M} = \mathrm{Lift}(M, \log)$. It's sufficient to show

$$D(\mathfrak{g}, Q) \otimes_{D(\mathfrak{g}, P)} (\tilde{M} \otimes V') = \tilde{M} \otimes i(V)' \qquad (1)$$

since can then apply $D(G) \otimes_{D(\mathfrak{g}, Q)} -$ to this isomorphism.

Fact: $i(V)' = D(\mathfrak{g}, Q) \otimes_{D(\mathfrak{g}, P)} V'$ as $D(\mathfrak{g}, Q)$-modules. $\quad (2)$

$$\begin{aligned}
&\mathrm{Hom}_{D(\mathfrak{g}, Q)}\left(D(\mathfrak{g}, Q) \otimes_{D(\mathfrak{g}, P)} (\tilde{M} \otimes V'), -\right) \\
&= \mathrm{Hom}_{D(\mathfrak{g}, P)}(\tilde{M} \otimes V', -) \\
&= \mathrm{Hom}_{D(\mathfrak{g}, P)}(V', \mathrm{Hom}(\tilde{M}, -)) \quad \text{adjunction} \\
&= \mathrm{Hom}_{D(\mathfrak{g}, Q)}(D(\mathfrak{g}, Q) \otimes_{D(\mathfrak{g}, P)} V', \mathrm{Hom}(\tilde{M}, -)) \\
&\overset{(2)}{=} \mathrm{Hom}_{D(\mathfrak{g}, Q)}(i(V)', \mathrm{Hom}(\tilde{M}, -)) \\
&= \mathrm{Hom}_{D(\mathfrak{g}, Q)}(\tilde{M} \otimes i(V)', -) \quad \text{adjunction}
\end{aligned}$$

So, by Yoneda's lemma, have (1) $\square$

Thanks!
Questions?