# Exponential at infinity

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Let $K$ be a spherically complete nonarchimedean field of mixed characteristic $(0,p)$ and $\mathcal{R}$ the Robba ring over $K$ with indeterminate $t$. Let $\partial$ denote differentiation with respect to $t$ on $\mathcal{R}$. Define $\omega := |p|^{1/(p-1)}$. Fix $\alpha \in K^\times$. Let $E$ be a free $\mathcal{R}$-module of rank 1 with generator $e$, regarded as a differential $\mathcal{R}$-module by setting

$$\partial e = (-\alpha/t^2)e.$$

Observe that $\exp(\alpha/t)$ is a formal solution for the differential equation $\partial + \alpha/t^2$. We know that $\exp(t)$ converges if and only if $|t| < \omega$, so $\exp(\alpha/t)$ converges if and only if

$$|\alpha/t| < \omega \iff |\alpha|\omega^{-1} < |t|.$$

Thus $E$ is a trivial differential $\mathcal{R}$-module if and only if $\exp(\alpha/t) \in \mathcal{R}$ if and only if $|\alpha| < \omega$.

Let us then consider the nontrivial case when $|\alpha| \geq \omega$. It follows from a straightforward insertion of some $\alpha$'s into the inductive calculations of [DMST13, theorem 1] that

$$\partial^n e = \left(\sum_{k=1}^{n} L(n,k)\alpha^k t^{-n-k}\right) e$$

where $L(n,k)$ denotes a Lah number

$$L(n,k) := \frac{n!}{k!}\binom{n-1}{k-1}.$$

In other words, we have

$$\partial^{[n]} e = \left(\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1}\alpha^k t^{-n-k}\right) e.$$

For $\rho < 1$, let us compute

$$R(E,\rho) = \min\left\{\rho, \liminf_{n\to\infty} \left|\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1}\alpha^k t^{-n-k}\right|_\rho^{-1/n}\right\}.$$

For fixed $n$, we have

$$\begin{aligned}
\left|\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1}\alpha^k t^{-n-k}\right|_\rho^{-1/n} &= \left(\max_k \left|\frac{1}{k!}\binom{n-1}{k-1}\right| |\alpha|^k \rho^{-n-k}\right)^{-1/n} \\
&= \min_k |k!|^{1/n} \left|\binom{n-1}{k-1}\right|^{-1/n} |\alpha|^{-k/n}\rho^{(n+k)/n} \\
&= \min_k \left|\binom{n-1}{k-1}\right|^{-1/n} \left(\frac{\omega\rho}{|\alpha|}\right)^{k/n} \rho\omega^{-\sigma(k)/n},
\end{aligned}$$

where we have used the fact that $|k!| = \omega^{k-\sigma(k)}$ in the last step.

Fixing $k = 1, \ldots, n$, observe that the binomial coefficient $\binom{n-1}{k-1}$ is an integer, so its norm is at most 1, so $\left|\binom{n-1}{k-1}\right|^{-1/n} \geq 1$. Also, we have

$$\omega|\alpha|^{-1}\rho < \omega|\alpha|^{-1} \leq 1,$$

which means that $(\omega|\alpha|^{-1}\rho)^{k/n} \geq \omega|\alpha|^{-1}\rho$. Finally, note that $\sigma(k) \geq 1$, so $\omega^{-\sigma(k)/n} \geq \omega^{-1/n}$. Putting these lower bounds together, we see that

$$\left|\binom{n-1}{k-1}\right|^{-1/n} \left(\frac{\omega\rho}{|\alpha|}\right)^{k/n} \rho\omega^{-\sigma(k)/n} \geq \omega^{1-(1/n)}|\alpha|^{-1}\rho^2$$

for all $k = 1, \ldots, n$, so

$$\liminf_{n\to\infty} \left|\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1}\alpha^k t^{-n-k}\right|_\rho^{-1/n} \geq \liminf_{n\to\infty} \omega^{1-(1/n)}|\alpha|^{-1}\rho^2 = \omega|\alpha|^{-1}\rho^2.$$

Now consider the subsequence given by terms of the form $n = p^s$ for $s \in \mathbb{N}$. For a fixed $n$ of this form, all three quantities

$$\left|\binom{n-1}{k-1}\right|^{-1/n}, (\omega|\alpha|^{-1}\rho)^{k/n}, \text{ and } \omega^{-\sigma(k)/n}$$

are minimized when $k = n$. Indeed, the first two quantities are always minimized when $k = n$ (even when $n$ is not of the form $p^s$), and the third is also minimized when $k = n$ since then $\sigma(k) = \sigma(n) = \sigma(p^s) = 1$. Thus

$$\left|\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1}\alpha^k t^{-n-k}\right|_\rho^{-1/n} = \omega^{1-(1/n)}|\alpha|^{-1}\rho^2.$$

Letting $s \to \infty$, the limit of this subsequence is $\omega|\alpha|^{-1}\rho^2$. Thus $\omega|\alpha|^{-1}\rho^2$ is both a subsequential limit as well as a lower bound for the limit inferior, so

$$\liminf_{n\to\infty} \left| \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} \alpha^k t^{-n-k} \right|_{\rho}^{-1/n} = \omega |\alpha|^{-1} \rho^2$$

which means that

$$R(E, \rho) = \min\{\rho, \omega|\alpha|^{-1}\rho^2\} = \omega|\alpha|^{-1}\rho^2.$$

Then

$$\lim_{\rho\to 1^-} R(E, \rho) = \omega|\alpha|^{-1}.$$

When $|\alpha| = \omega$, we see that we have $R(E, \rho) = \rho^2$, so $E$ is overconvergent and of highest slope $\beta = 1$ [CM00, 4.2–2]. This slope $\beta$ is also the irregularity of $E$, since $E$ is of rank 1.

## References


[CM00] Gilles Christol and Zoghman Mebkhout. Sur le théorème de l'indice des équations différentielles $p$-adiques. III. *Annals of Mathematics. Second Series*, 151(2):385–457, 2000. doi:10.2307/121041.

[DMST13] Siad Daboul, Jan Mangaldan, Michael Z. Spivey, and Peter J. Taylor. The Lah numbers and the $n$th derivative of $e^{1/x}$. *Mathematics Magazine*, 86(1):39–47, 2013. URL: http://mathcs.pugetsound.edu/~mspivey/Exp.pdf, doi:10.4169/math.mag.86.1.039.