Cotangent complex

Contents

1 Introduction

This is an attempt at a gentle introduction to the cotangent complex. Recall that whenever we have morphisms of schemes

$$Y \xrightarrow{f} X \longrightarrow S,$$

there is an exact sequence

$$f^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} \longrightarrow 0 \tag{1}$$

of $\mathcal{O}_Y$-modules [3, tag 01UX]. It isn't always exact on the left.

Example 1.1. Let S be the spectrum of a field k of characteristic not equal to 2 or 3. Let $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ be the cusp and let $Y = \operatorname{Spec} k[t]$, with f the "normalization of the cusp," induced by the unique k-algebra homomorphism such that $x \mapsto t^2$ and $y \mapsto t^3$. On global sections, (1) induces the following exact sequence of $k[t]$-modules.

$$(k[t]dx \oplus k[t]dy)/(2t^3dy - 3t^4dx) \longrightarrow k[t]dt \longrightarrow k[t]dt/(tdt) \longrightarrow 0$$

The left-most map is given by $dx \mapsto 2tdt$ and $dy \mapsto 3t^2dt$, and it is easily seen that $\omega = 3tdx - 2dy$ is a nonzero element of the kernel (in fact, it can be verified that ω generates the kernel). Thus, we see that (1) is not exact on the left.

Sometimes, even when (1) is exact on the left, it doesn't stay exact on the left after tensoring with some $\mathcal{O}_Y$-module and this lack of exactness reflects some geometry.

Example 1.2. Suppose S is the spectrum of a field k of characteristic not 2. Let $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[x, y]/(y^2 - x)$ with $f : Y \to X$ the morphism induced by the inclusion $k[x] \hookrightarrow k[x, y]/(y^2 - x)$. Then taking global sections in (1) gives us the following sequence of modules over $B := k[x, y]/(y^2 - x)$.

$$Bdx \longrightarrow (Bdx \oplus Bdy)/(2ydy - dx) \longrightarrow Bdy/(2ydy) \longrightarrow 0$$

It is not hard to see that the first map is injective. If $P \in Y$ is the closed point corresponding to the maximal ideal $(x - a^2, y - a) \subset B$, then tensoring up to the residue field $\kappa(P) = B/(x - a^2, y - a)$ gives us the following sequence of modules over $\kappa(P)$.

$$\kappa(P)dx \longrightarrow (\kappa(P)dx \oplus \kappa(P)dy)/(2ady - dx) \longrightarrow \kappa(P)dy/(2ady) \longrightarrow 0$$

If $a \neq 0$, then the map on the left is evidently injective—it is even an isomorphism. But when $a = 0$, the map becomes zero and we lose exactness on the left. More precisely, there are nonzero differential forms near the origin in X which vanish after pulling back to Y and evaluating at the origin in Y. So the loss of exactness on the left is reflecting a lack of smoothness over the origin in X.

Now when f is a closed embedding defined by a quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_X$, we know that $\Omega^1_{Y/X} = 0$, but we can extend this sequence one term to the left and get an exact sequence

$$f^*\mathcal{I} \xrightarrow{\delta} f^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow 0 \tag{2}$$

of $\mathcal{O}_Y$-modules, where if $d : \mathcal{O}_X \to \Omega^1_{X/S}$ is the universal derivation, then δ is the unique $\mathcal{O}_Y$-linear map given locally by $\delta(s \otimes 1) = ds \otimes 1$ for $s \in \mathcal{I}$ [3, tag 01UZ]. Again, this isn't always exact on the left.

Example 1.3. Suppose S is the spectrum of a field k. Let $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[x]/(x^2)$ with f induced by the surjection $k[x] \twoheadrightarrow k[x]/(x^2)$. Then taking global sections in (2) gives us the following sequence of modules over $B := k[x]/(x^2)$.

$$(x^2)/(x^4) \longrightarrow Bdx \longrightarrow Bdx/(2xdx) \longrightarrow 0$$

Then $x^3 \in (x^2)/(x^4)$ is nonzero and maps to $3x^2dx = 0$ in Bdx, so this map is not injective. In characteristic 2, the left-most map vanishes altogether.

All of this discussion of failure of exactness on the left suggests that there might be a geometrically meaningful long exact sequence hiding somewhere.

2 Cotangent complex

For every morphism $X \to S$ of schemes, we assign a complex $L_{X/S}$ of locally free $\mathcal{O}_X$-modules in non-positive degrees called the *cotangent complex* of X over S. We will avoid a construction, but here are some of the important properties it satisfies.

(CC1) (Functoriality and base change) Given a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

there is a natural homomorphism

$$v^*L_{X/S} \longrightarrow L_{X'/S'}.$$

of complexes of $\mathcal{O}_{X'}$-modules. If the square is cartesian and $\operatorname{Tor}^i_{\mathcal{O}_S}(\mathcal{O}_{S'}, \mathcal{O}_X) = 0$ for all $i \neq 0$, then this homomorphism is a quasi-isomorphism.

(CC2) (Fundamental triangle) For every triple

$$Y \xrightarrow{f} X \longrightarrow S$$

of morphisms of schemes, there is a natural distinguished triangle

$$f^*L_{X/S} \longrightarrow L_{Y/S} \longrightarrow L_{Y/X} \longrightarrow f^*L_{X/S}[1]$$

in the derived category $\mathcal{D}(Y)$ of $\mathcal{O}_Y$-modules.

(CC3) For every morphism $X \to S$, there is a homomorphism $\gamma : L_{X/S} \to \Omega^1_{X/S}[0]$ inducing an isomorphism $H^0(L_{X/S}) \to \Omega^1_{X/S}$. Moreover, γ is a quasi-isomorphism when $X \to S$ is smooth.

Remark 2.1. If you don't like the derived category, the "practical" consequence of (CC2) is the existence of the following long exact sequence.

$$\cdots \to H^{-1}(f^*L_{X/S}) \longrightarrow H^{-1}(L_{Y/S}) \longrightarrow H^{-1}(L_{Y/X}) \to f^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} \longrightarrow 0$$

Really though, the derived category is your friend. There's no reason you should dislike it.

Example 2.2. Continuing with example 1.1, note that $Y \to S$ is smooth, so $H^i(L_{Y/S}) = 0$ for all $i \lneq 0$. Thus $H^{-1}(L_{Y/X})$ gets identified with the kernel of $f^*\Omega^1_{X/S} \to \Omega^1_{Y/S}$.

Recall that a morphism of schemes $f : X \to S$ is a *locally complete intersection* if there is a cover of X by open subsets $U \subset X$ such that $U \to S$ factors as

$$\begin{array}{ccc} U & \hookrightarrow & P \\ & \searrow & \downarrow \\ & & S \end{array}$$

where $P \to S$ is smooth and $U \hookrightarrow P$ is a closed embedding whose ideal $\mathcal{I} \subset \mathcal{O}_P$ is generated by a regular sequence of sections on P [3, tag 068E].

Proposition 2.3. *If $X \to S$ is a locally complete intersection, the natural map $L_{X/S} \to \tau_{\geq -1}L_{X/S}$ is a quasi-isomorphism.*

Proof. The statement is local on X, so using (CC1) we may assume that S is affine and that $X \to S$ itself factors through a closed embedding $i : X \hookrightarrow P$ over S with P a smooth S-scheme. Now $L_{P/S} \to \Omega^1_{P/S}[0]$ is a quasi-isomorphism by (CC3), so the distinguished triangle

$$i^*L_{P/S} \longrightarrow L_{X/S} \longrightarrow L_{X/P} \longrightarrow i^*L_{P/S}[1]$$

of (CC2) induces isomorphisms $H^i(L_{X/S}) \to H^i(L_{X/P})$ for all $i \lneq -1$. In other words, we reduce to the case when $P = S = \operatorname{Spec} A$ and $X = \operatorname{Spec} A/I$ for $I \subset A$ generated by a regular sequence $(f_1, \dots, f_r)$. We can inductively reduce to the case $r = 1$. Indeed, let $X' = \operatorname{Spec} A/(f_1, \dots, f_{r-1})$, so that $X \hookrightarrow S$ factors through a closed embedding $i : X \hookrightarrow X'$. Then we have a distinguished triangle

$$i^*L_{X'/S} \longrightarrow L_{X/S} \longrightarrow L_{X/X'} \longrightarrow i^*L_{X'/S}[1]$$

from (CC2) and the ideal of $X' \hookrightarrow S$ is generated by a regular sequence of length $r - 1$ and the ideal of $X \hookrightarrow X'$ by a regular sequence of length 1. Vanishing of cohomology outside $[-1, 0]$ for these two therefore implies the same for $X \hookrightarrow S$.

Thus we are now in the situation where $X = \operatorname{Spec} A/(f)$ for $f \in A$ not a zero-divisor. Consider the following diagram where $\operatorname{Spec}\mathbb{Z} \hookrightarrow \operatorname{Spec}\mathbb{Z}[T]$ is given by $T \mapsto 0$ and $\operatorname{Spec} A \to \operatorname{Spec}\mathbb{Z}[T]$ is given by $T \mapsto f$.

$$\begin{array}{ccc} \operatorname{Spec} A/(f) & \longrightarrow & \operatorname{Spec}\mathbb{Z} \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec}\mathbb{Z}[T] \end{array}$$

Note that $\mathbb{Z}$ can be resolved as a $\mathbb{Z}[T]$-module by the two-term complex of $\mathbb{Z}[T]$-modules

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T]$$

where the nontrivial differential is $1 \mapsto T$. Applying $- \otimes_{\mathbb{Z}[T]} A$ then gives us the two-term complex of A-modules

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow A$$

where the nontrivial differential is $1 \mapsto f$. This shows that

$$\mathbb{Z} \otimes_{\mathbb{Z}[T]} A = \operatorname{coker}(\cdot f : A \to A) = A/(f)$$

so that the square is cartesian, and also that

$$\operatorname{Tor}^{-1}_{\mathbb{Z}[T]}(\mathbb{Z}, A) = \ker(\cdot f : A \to A) = 0$$

since f is not a zero divisor in A. Thus we have $\operatorname{Tor}^{i}_{\mathbb{Z}[T]}(\mathbb{Z}, A) = 0$ for all $i \neq 0$, so using (CC1), we reduce to the case $S = \operatorname{Spec}\mathbb{Z}[T]$ and $X = \operatorname{Spec}\mathbb{Z}$ with $X \hookrightarrow S$ induced by $T \mapsto 0$. Considering $i : X \hookrightarrow S$ as a morphism of schemes over $X = \operatorname{Spec}\mathbb{Z}$, we apply (CC2) to get a distinguished triangle

$$i^*L_{S/X} \longrightarrow L_{X/X} \longrightarrow L_{X/S} \longrightarrow i^*L_{S/X}[1].$$

Now (CC3) implies that $L_{X/X} = 0$ and also that $L_{S/X} = \Omega^1_{X/S}[0]$. The associated long exact sequence shows that $H^i(L_{X/S}) = 0$ for all other $i \lneq -1$, and this is precisely what we wanted to show. □

Proposition 2.4 ([3, tag 08R6]). *Given morphisms of schemes*

$$Y \xhookrightarrow{\ i\ } X \longrightarrow S$$

with i a closed embedding with ideal $\mathcal{I} \subset \mathcal{O}_X$ and $X \to S$ smooth, consider the map $\delta : i^*\mathcal{I} \to i^*\Omega^1_{X/S}$ from (2) as a complex $NL_{Y/S}$ of $\mathcal{O}_Y$-modules concentrated in degrees -1 and 0. Then there is a natural homomorphism $\kappa : L_{Y/S} \to NL_{Y/S}$ of complexes which induces a quasi-isomorphism $\tau_{\geq -1}L_{Y/S} \to NL_{Y/S}$.

Remark 2.5. Keeping the notation of proposition 2.4, the conormal exact sequence (2) furnishes an isomorphism $H^0(NL_{Y/S}) \to \Omega^1_{Y/S}$. Precomposing with $H^0(\kappa)$ gives us an isomorphism $H^0(L_{Y/S}) \to \Omega^1_{Y/S}$. One can verify that this coincides with the isomorphism of (CC3) above.

Example 2.6. Let us make explicit what is going on locally, on the level of rings. Let S be the spectrum of a ring A, X the spectrum of $P = A[x_1, \ldots, x_n]$, and $I \subset P$ an ideal. Set $B = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B$. Then $X \to S$ is smooth and $Y \hookrightarrow X$ is a closed immersion, so $NL_{Y/S}$ is given on global sections by

$$\cdots \longrightarrow 0 \longrightarrow I \otimes_P B \longrightarrow \Omega^1_{P/A} \otimes_P B$$

where the non-trivial differential is the unique B-linear one given by $f \otimes 1 \mapsto df \otimes 1$. When I is generated by a regular sequence, propositions 2.4 and 2.3 together tell us that this complex is quasi-isomorphic to the cotangent complex $L_{Y/S}$.

Corollary 2.7. *If $i : Y \hookrightarrow X$ is a closed embedding with ideal $\mathcal{I} \subset \mathcal{O}_X$, then $H^{-1}(L_{Y/X}) = i^*\mathcal{I}$.*

Proof. We apply proposition 2.4 with $X \to S$ the identity map on X. Then $i^*\Omega^1_{X/X} = 0$, so $NL_{Y/X} = i^*\mathcal{I}[1]$, which means that $H^{-1}(L_{Y/X}) = i^*\mathcal{I}$. □

Remark 2.8. If $X \to S$ is any morphism of schemes and $i : Y \hookrightarrow X$ is a closed embedding, we can take the distinguished triangle (CC2) associated to the composite $Y \hookrightarrow X \to S$. We get a long exact sequence as in remark 2.1, and we make the substitution $H^{-1}(L_{Y/X}) = i^*\mathcal{I}$.

$$\cdots \to H^{-1}(i^*L_{X/S}) \longrightarrow H^{-1}(L_{Y/S}) \longrightarrow i^*\mathcal{I} \to i^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

One can verify that the connecting map $i^*\mathcal{I} \to i^*\Omega^1_{X/S}$ is precisely the map δ in (2).

Example 2.9. Consider again the situation of example 1.3. Then $X \to S$ is smooth, so $H^{-1}(i^*L_{X/S}) = 0$. Thus $H^{-1}(L_{Y/S}) = \ker\delta$. Taking global sections, this corresponds to the $k[x]/(x^2)$-module $(x^3)/(x^4)$.

Recall that a complex of $\mathcal{O}_X$-modules on a scheme X is *perfect* if it is locally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank.

Corollary 2.10. *If $X \to S$ is a locally complete intersection, then $\tau_{\geq -1}L_{X/S}$ is a perfect complex of $\mathcal{O}_X$-modules.*

Proof. The statement is local on X, so we may $X \to S$ factors through a closed embedding $X \hookrightarrow P$ over S with P a smooth S-scheme and such that the ideal $\mathcal{I} \subset \mathcal{O}_P$ of $X \hookrightarrow P$ is generated by a regular sequence of global sections. Proposition 2.4 furnishes a quasi-isomorphism $\tau_{\geq -1}L_{X/S} \to NL_{X/S}$. We now need two facts from commutative algebra: that $\Omega^1_{P/S}$ is a locally free $\mathcal{O}_P$-module of finite rank since $P \to S$ is smooth [3, tag 00TH], and that $i^*\mathcal{I}$ is a locally free $\mathcal{O}_X$-module of finite rank since $\mathcal{I}$ is generated by a regular sequence [3, tag 07CU]. Thus $NL_{X/S}$ is a complex of locally free $\mathcal{O}_X$-modules of finite rank, so $\tau_{\geq -1}L_{X/S}$ is perfect. □

3 Flat dimension

Suppose we are given a morphism of schemes $X \to S$. Example 1.2 showed us that it is not just $H^i(L_{X/S})$ that gives us geometric information, but also $H^i(L_{X/S} \otimes F)$ for various $\mathcal{O}_X$-modules F. Recall that a complex E of $\mathcal{O}_X$-modules has *flat dimension* at most n if $H^i(E \otimes^{\mathbb{L}} F) = 0$ for all $i \notin [-n, 0]$ and all $\mathcal{O}_X$-modules F.

Example 3.1. When $X \to S$ is smooth, $\Omega^1_{X/S}$ is a locally free $\mathcal{O}_X$-module. Thus (CC3) implies that $L_{X/S}$ has flat dimension 0. More generally, when $X \to S$ is a locally complete intersection, we know that $L_{X/S} \to \tau_{\geq -1}L_{X/S}$ is a quasi-isomorphism by proposition 2.3. But $\tau_{\geq -1}L_{X/S}$ is perfect by corollary 2.10 and so evidently has flat dimension at most 1, so $L_{X/S}$ also has flat dimension at most 1.

Remark 3.2 ([1, section 1])**.** The cotangent complex is remarkably well-behaved. Suppose $f : X \to S$ is a morphism of locally noetherian schemes.

- (First vanishing theorem) The morphism f has geometrically regular fibers if and only if $L_{X/S}$ has flat dimension 0. Recall that a f is smooth if and only if it is flat, locally of finite type and has geometrically regular fibers. Thus, having flat dimension 0 is very close to being smooth.

- (Second vanishing theorem) The morphism f is a locally complete intersection if and only if $L_{X/S}$ has flat dimension at most 1.
- (Conjecture) The morphism f is a locally complete intersection if and only if it has locally finite flat dimension and $L_{X/S}$ has finite flat dimension. This conjecture has been settled when S is a $\mathbb{Q}$-scheme and in a wide assortment of positive characteristic situations as well.
- (Conjecture) If $L_{X/S}$ has finite flat dimension, then it has flat dimension at most 2.

Given this last conjecture, we see that it would be useful to be able to calculate $\tau_{\geq -2}L_{X/S}$ explicitly. This is done by a construction of Lichtenbaum and Schlessinger [3, tag 09AM].

Example 3.3. Consider the normalization of the cusp from example 1.1 again, and let F be an $\mathcal{O}_Y$-module. The distinguished triangle

$$f^*L_{X/S} \otimes F \longrightarrow L_{Y/S} \otimes F \longrightarrow L_{Y/X} \otimes F \longrightarrow (f^*L_{X/S} \otimes F)[1]$$

induces a long exact sequence on homology. Since $Y \to S$ is smooth, we know that $H^i(L_{Y/S} \otimes F) = 0$ for all $i \lneq 0$, so we get isomorphisms $H^{i-1}(L_{Y/X} \otimes F) \to H^i(f^*L_{X/S} \otimes F)$ for all $i \leq -1$. Now $X \to S$ is a locally complete intersection, so $L_{X/S}$ has flat dimension at most 1, so $f^*L_{X/S}$ does as well [3, tag 066L]. The isomorphisms above therefore imply that $L_{Y/X}$ has flat dimension at most 2. Since $X \to S$ is not smooth, the first vanishing theorem leads us to expect that $L_{X/S}$ should have nonzero flat dimension, so there should be an $\mathcal{O}_Y$-module F such that $H^{-1}(f^*L_{X/S} \otimes F) \neq 0$. This would imply that $H^{-2}(L_{Y/X} \otimes F) \neq 0$, and then the second vanishing theorem should in turn imply that $Y \to X$ is not a locally complete intersection. Let us check these two expectations explicitly.

First, let us produce an $\mathcal{O}_Y$-module F such that $H^{-1}(f^*L_{X/S} \otimes F) \neq 0$. Let $O \in X$ be the origin and $\kappa(O)$ its residue field, regarded as a skyscraper sheaf. Using the explicit description of 2.4, we compute that $H^{-1}(L_{X/S} \otimes \kappa(O)) \neq 0$. Now note that f is flat [2, chapter III, proposition 9.7], so if we let $F = f^*\kappa(O)$, then

$$H^{-1}(f^*L_{X/S} \otimes F) = H^{-1}(f^*(L_{X/S} \otimes \kappa(O))) = f^*H^{-1}(L_{X/S} \otimes \kappa(O)) \neq 0.$$

Second, let us show that $Y \to X$ is not a locally complete intersection. Note that if $A = k[x, y]/(y^2 - x^3)$, then $k[t] = A[t]/(t^2 - x, t^3 - y)$. But $(t^2 - x, t^3 - y)$ is not a regular sequence. Indeed,

$$A[t]/(t^2 - x) = k[x, y, t]/(y^2 - x^3, t^2 - x) = k[y, t]/(y^2 - t^6)$$

and $y^2 - t^6 = (y - t^3)(y + t^3)$, so $t^3 - y$ is a zero-divisor in this ring.

4 Deformation theory

Let $S \hookrightarrow S'$ be a first order thickening and $f : X \to S$ be morphism of schemes. A *deformation* of X is a first order thickening $X \hookrightarrow X'$ and a flat morphism $X' \to S'$ such that the following square is commutative and cartesian.

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \hookrightarrow & S' \end{array} \tag{3}$$

Remark 4.1. In the literature, the requirements that the square be cartesian and $X' \to S'$ be flat are often rephrased in a slick way. Suppose we are given a commutative square of first order thickenings of schemes as in (3). Let $\mathcal{I}$ be the ideal of $S \hookrightarrow S'$ and $\mathcal{J}$ the ideal of $X \hookrightarrow X'$. One can construct a certain canonical homomorphism $f^*\mathcal{I} \to \mathcal{J}$ of $\mathcal{O}_X$-modules using commutativity of the square and the fact that $\mathcal{I}$ and $\mathcal{J}$ are square-zero. Surjectivity of $f^*\mathcal{I} \to \mathcal{J}$ is equivalent to the commutative square being cartesian [3, tag 08L2]. Moreover, when $f^*\mathcal{I} \to \mathcal{J}$ is surjective, the morphism $X' \to S'$ is flat if and only if $f^*\mathcal{I} \to \mathcal{J}$ is an isomorphism [3, tag 08L1].

Theorem 4.2 ([3, tag 08UZ])**.** *Let $S \hookrightarrow S'$ be a first order thickening with ideal $\mathcal{I} \subset \mathcal{O}_{S'}$ and $f : X \to S$ a morphism of schemes.*

(a) There exists a canonical element $o(f) \in \operatorname{Ext}^2_X(L_{X/S}, f^*\mathcal{I})$, called the obstruction class*, whose vanishing is a necessary and sufficient condition for there to exist a deformation of X.*

(b) The set of isomorphism classes of all deformations of X is a pseudo-torsor for $\operatorname{Ext}^1_X(L_{X/S}, f^*\mathcal{I})$.[1]

(c) Given any deformation X' of X, the automorphisms of X' as a deformation of X are naturally in bijection with $\operatorname{Ext}^0_X(L_{X/S}, f^*\mathcal{I}) = \operatorname{Hom}_X(\Omega^1_{X/S}, f^*\mathcal{I})$.

Remark 4.3. We can precisely describe the bijection in part (c). Note that $\operatorname{Hom}_X(\Omega^1_{X/S}, f^*\mathcal{I}) = \operatorname{Der}_{f^{-1}\mathcal{O}_S}(\mathcal{O}_X, f^*\mathcal{I})$. Given a $f^{-1}\mathcal{O}_S$-linear derivation $d : \mathcal{O}_X \to f^*\mathcal{I}$, recall that we have a natural isomorphism $f^*\mathcal{I} \to \ker(j^\sharp : \mathcal{O}_{X'} \twoheadrightarrow \mathcal{O}_X)$, so we can regard d as a map $\mathcal{O}_X \to \mathcal{O}_{X'}$. We then send d to the automorphism of X' that is the identity on topological spaces, and whose map $\mathcal{O}_{X'} \to \mathcal{O}_{X'}$ on sheaves of rings is $1 + d \circ j^\sharp$.

Suppose S is a locally noetherian scheme and $f : X \to S$ is locally of finite type. Then $L_{X/S}$ is quasi-isomorphic to a bounded above complex of locally free $\mathcal{O}_X$-modules of finite rank, so

$$\underline{\operatorname{Ext}}^n_X(L_{X/S}, f^*\mathcal{I}) = H^n(\underline{\operatorname{Hom}}_X(L_{X/S}, f^*\mathcal{I})).$$

If X is affine, the local-to-global spectral sequence for ext degenerates and we have

$$\operatorname{Ext}^n_X(L_{X/S}, f^*\mathcal{I}) = H^n(\operatorname{Hom}_X(L_{X/S}, f^*\mathcal{I})).$$

If $f : X \to S$ is smooth, then $L_{X/S} \to \Omega^1_{X/S}[0]$ is a quasi-isomorphism, so

$$\operatorname{Ext}^n_X(L_{X/S}, f^*\mathcal{I}) = H^n(\operatorname{Hom}_X(\Omega^1_{X/S}, f^*\mathcal{I}))$$

is non-zero only for $n = 0$. Thus smooth affine schemes always have a unique deformation, but that deformation can have non-trivial automorphisms. More generally, if $f : X \to S$ is a locally complete intersection, then

$$\operatorname{Ext}^n_X(L_{X/S}, f^*\mathcal{I}) = H^n(\operatorname{Hom}_X(\tau_{\geq -1}L_{X/S}, f^*\mathcal{I}))$$

is non-zero only for $n = 0$ and 1. Thus affine locally complete intersections always have deformations, but the deformations may be non-unique and each one may have non-trivial automorphisms. Finally, general affine schemes may or may not have any deformations. Let us consider examples of each of these phenomena. The following calculations have been done here like a caveman—I have not been careful to remember isomorphisms.

Example 4.4. Suppose k is a field and $S \hookrightarrow S'$ is the first order thickening corresponding to the surjective ring homomorphism $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ given by $\varepsilon \mapsto 0$. Let $X = \operatorname{Spec} k[t]$ be the affine line over S. Then $X' = \operatorname{Spec} k[\varepsilon, t]/(\varepsilon^2)$ is the unique deformation of X, but it has automorphisms $\sigma : X' \to X'$ given by $t \mapsto t + \varepsilon s$ for some $s \in k[\varepsilon, t]/(\varepsilon^2)$. But any ε appearing in s will be killed by the ε in front, so in fact we can take $s \in k[t] \subset k[\varepsilon, t]/(\varepsilon^2)$. So the automorphisms of X' are in bijection with $k[t]$. Notice also that $\Omega^1_{X/S}$ corresponds to the free $k[t]$-module $k[t]dt$, and $f^*\mathcal{I}$ to $(\varepsilon) \otimes_{k[\varepsilon]/\varepsilon^2} k[t] \simeq k[t]$, so

$$\operatorname{Hom}_{k[t]}(k[t]dt, (\varepsilon) \otimes_{k[\varepsilon]/\varepsilon^2} k[t]) \simeq k[t]$$

also.

Example 4.5 (Deformations of the node)**.** Again, suppose k is a field and $S \hookrightarrow S'$ is the first order thickening corresponding to the surjective ring homomorphism $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ given by $\varepsilon \mapsto 0$ and let $X = \operatorname{Spec} k[x, y]/(xy)$. Then $X'_a := \operatorname{Spec} k[x, y, \varepsilon]/(\varepsilon^2, xy - a\varepsilon)$ is certainly a deformation of X for any $a \in k$. Any isomorphism $X'_0 \to X'_a$ would have to be given by $x \mapsto x - \varepsilon s$ and $y \mapsto y - \varepsilon t$ for $s, t \in k[x, y, \varepsilon]/(\varepsilon^2, xy)$, and we can assume ε does not appear in s or t. But then

$$xy - a\varepsilon \mapsto xy - \varepsilon(ys + xt + a)$$

[1] A set S is a *pseudo-torsor* for a group G if G acts simply transitively on S. Then S is a *torsor* if, in addition, it is nonempty.

which vanishes modulo xy precisely if $ys + xt = -a$ modulo xy. A little thought shows that this is only possible if s is a multiple of x, t a muliply of y and $a = 0$. The multiples of x in $k[x, y, \varepsilon]/(\varepsilon^2, xy)$ are evidently just $k[x]$ and similarly with y, so we see that the automorphisms of X_0 are $k[x] \oplus k[y]$ and that X_a is not isomorphic to X_0 for any $a \in k$. Generalizing slightly, we can show that X'_a is not isomorphic to X'_b for any $a \neq b$ in k.

Let's now compare this to ext modules of the cotangent complex. Let $B = k[x, y]/(xy)$ Note that $f^*\mathcal{I}$ corresponds to the B-module $(\varepsilon) \otimes_{k[\varepsilon]/(\varepsilon^2)} B \simeq B$ and $NL_{X/S}$ to the complex of B-modules

$$\cdots \longrightarrow 0 \longrightarrow (xy) \otimes_{k[x,y]} B \longrightarrow Bdx \oplus Bdy$$

where the non-trivial differential is given by $xy \otimes f \mapsto fydx + fxdy$. Then $\mathrm{Hom}_X(NL_{X/S}, f^*\mathcal{I})$ is the complex

$$\mathrm{Hom}_B(Bdx \oplus Bdy, B) \longrightarrow \mathrm{Hom}((xy) \otimes_{k[x,y]} B, B) \longrightarrow 0 \longrightarrow \cdots$$

which is isomorphic to the complex

$$B \oplus B \xrightarrow{(y,x)} B \longrightarrow 0 \longrightarrow \cdots .$$

Now homology in degree 1 of this complex is $B/(x, y) \simeq k$ so the set of deformations is in bijection with k, as we would hope. Moreover, the automorphisms of any given deformation are in bijection with $xB \oplus yB \simeq k[x] \oplus k[y]$, which again we had observed earlier.

Example 4.6 (Obstructed deformations)**.**

References

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